Example 3

\[ V^+ = +24 \text{ V} \]
\[ V^- = -24 \text{ V} \]
\[ \beta = 99 \]
\[ \alpha = 0.99 \]
\[ R_C = 7.5 \text{ k}\Omega \]
\[ V_{BE} = 0.65 \text{ V} \]
\[ R_1 = 2 \text{ M}\Omega \]
\[ R_2 = 1.2 \text{ M}\Omega \]
\[ R_E = 6.2 \text{ k}\Omega \]

\[ V_{BB} = V^+ \frac{R_2}{R_C + R_1 + R_2} + V^- \frac{R_1 + R_C}{R_C + R_1 + R_2} - I_C \frac{R_C}{R_C + R_1 + R_C} R_2 \]

\[ = -6.042 - 2806 I_C \]

\[ R_{BB} = (R_1 + R_C) || R_2 = 751.1 \text{ k}\Omega \]

\[ V_{BB} - V_{EE} = -6.042 - 2806 I_C - (-24) \]

\[ = 17.958 - 2806 I_C \]
But \( V_{BB} - V_{EE} = I_B R_{BB} + V_{BE} + I_E R_E \)

\[ = \frac{I_C}{\beta} R_{BB} + V_{BE} + \frac{I_C}{\alpha} R_E \]

\[ = 13.85 \times 10^3 I_C + 0.65 \]

\[ \Rightarrow 17.958 - 28.06 I_C = 13.85 \times 10^3 I_C + 0.65 \]

\[ \Rightarrow I_C = \frac{17.958 - 0.65}{28.06 + 13.85 \times 10^3} = 1.039 \text{ mA} \]

Next we test for the active mode.

\[ V_{CB} = (V_{CC} - I_C R_{CC}) - (V_{BE} + I_E R_E + V^-) \]

But \( V_{CC} = V^+ \frac{R_1 + R_2}{R_C + R_1 + R_2} + V^- \frac{R_C}{R_C + R_1 + R_2} \)

\[ - I_B \frac{R_2}{R_C + R_1 + R_2} \]

\[ = 23.858 \text{ V,} \]

\( R_{CC} = R_C \| (R_1 + R_2) = 74.82 \Omega \)

\[ V_{CB} = (V_{CC} - I_C R_{CC}) - (V_{BE} + \frac{I_C}{\alpha} R_E + V^-) \]

\[ = 32.92 \text{ V} \Rightarrow \text{active mode} \]
The Small-Signal \( \pi \) Model of the BJT

We assume the active mode and neglect the leakage current terms.

\[ I_C = I_S \, e^{V_{BE}/V_T} \]

\[ I_B = \frac{1}{\beta} \, I_C \]

\[ I_S = I_{So} \left( 1 + \frac{V_{CE}}{V_A} \right) \]

\[ \beta = \beta_0 \left( 1 + \frac{V_{CE}}{V_A} \right) \]

First, we look at the transfer characteristics for \( V_{CE} = \text{const.} \Rightarrow I_S = \text{const.} \Rightarrow I_C \) is a function of \( V_{BE} \) only.
The Q point is \((V_{BE}, I_c)\). We wish to relate the small-signal change in \(I_c\) to the small-signal change in \(V_{BE}\). The slope of the curve at the Q point is used to do this.

\[ I_c = I_s e^{\frac{V_{BE}}{V_T}} \]

\[ \frac{dI_c}{dV_{BE}} = \frac{1}{V_T} I_c e^{\frac{V_{BE}}{V_T}} = \frac{I_c}{V_T} \]

We define the transconductance \(g_m\) by

\[ g_m = \frac{dI_c}{dV_{BE}} = \frac{I_c}{V_T} \]
This must be equal to

\[ g_m = \frac{\Delta \dot{I}_C}{\Delta V_{BE}} = \frac{\dot{I}_C}{V_{be}} \]

\[ \Rightarrow \dot{I}_C = g_m V_{be} \]

This only holds for \( V_{ce} = \text{const.} \)

Now suppose we hold \( V_{be} = \text{const.} \) and vary \( V_{ce} \). We have already seen that if \( V_{BE} = \text{const.} \), then \( \dot{I}_B = \text{const.} \)

\[ \Rightarrow \dot{I}_C = \beta \dot{I}_B = \beta_0 \left( 1 + \frac{V_{CE}}{V_A} \right) \dot{I}_B \]

We plot the output characteristics with \( \dot{I}_B = \text{const.} \) as follows:

![Graph of output characteristics](image-url)
\[ I_c = \beta_0 \left( 1 + \frac{V_{CE}}{V_A} \right) I_B \]
\[ \frac{dI_c}{dV_{CE}} = \beta_0 \frac{1}{V_A} I_B \]

But \[ \beta_0 I_B = \frac{I_C}{1 + \frac{V_{CE}}{V_A}} \]

\[ \Rightarrow \frac{dI_c}{dV_{CE}} = \frac{1}{V_A} \frac{I_C}{1 + \frac{V_{CE}}{V_A}} = \frac{I_C}{V_A + V_{CE}} \]

Let us define the resistance \( R_0 \) by

\[ \frac{1}{R_0} = \frac{dI_c}{dV_{CE}} = \frac{I_C}{V_A + V_{CE}} \]

\[ \Rightarrow R_0 = \frac{V_A + V_{CE}}{I_C} \]

We can relate a small-signal change in \( I_c \) to a small-signal change in \( V_{CE} \) as follows

\[ \frac{1}{R_0} = \frac{\Delta I_c}{\Delta V_{CE}} = \frac{I_c}{V_{CE}} \]

\[ \Rightarrow I_c = \frac{V_{CE}}{R_0} \]
This only holds for $V_{BE} = \text{const.}$ (or $I_B = \text{const.}$). Suppose both $V_{BE}$ and $V_{CE}$ vary. Let $V_{BE} = \Delta V_{BE}$, $V_{CE} = \Delta V_{CE}$, and $I_C = \Delta I_C$. It follows that

$$I_C = q_m V_{be} + \frac{V_{CE}}{N_0}$$

Now, let us plot $I_B$ versus $V_{BE}$ for $V_{CE} = \text{const.}$

$$I_B = \frac{1}{\beta} I_C = \frac{I_{S}}{\beta} e^{rac{V_{BE}}{V_T}} = \frac{I_{S0}}{\beta_0} e^{rac{V_{BE}}{V_T}}$$

Note that the $1 + \frac{V_{CE}}{V_A}$ terms in $I_S$ and $\beta$ have cancelled.
The Q point is \((v_{BE}, I_B)\). We wish to relate the small-signal change in \(I_B\) to the small-signal change in \(V_{BE}\). The slope of the curve at the Q point is used to do this:

\[
I_B = \frac{I_{so}}{\beta_0} e^{V_{BE}/V_T}
\]

\[
\frac{dI_B}{dV_{BE}} = \frac{1}{V_T} \frac{I_{so}}{\beta_0} e^{V_{BE}/V_T}
\]

\[
= \frac{I_B}{V_T}
\]

Let us define the resistance \(r_T\) by:

\[
\frac{1}{r_T} = \frac{dI_B}{dV_{BE}} = \frac{I_B}{V_T}
\]

\[
\Rightarrow \quad r_T = \frac{V_T}{I_B}
\]

We can relate a small-signal change in \(I_B\) to a small-signal change in
$V_{be}$ as follows

\[ \frac{1}{R_{\pi}} = \frac{\Delta \lambda_B}{\Delta V_{BE}} = \frac{\lambda_B}{V_{be}} \]

\[ \Rightarrow \lambda_B = \frac{V_{be}}{R_{\pi}} \]

Notice that $\lambda_B$ is not a function of $V_C$ because the Early effect cancels in the equation for $\lambda_B$.

Summary:

\[ \lambda_B = \frac{V_{be}}{R_{\pi}} \]

\[ \lambda_C = g_m V_{be} + \frac{V_{ce}}{R_0} \]

Let us define $\lambda'_C = g_m V_{be}$ so that $\lambda_C$ can be written

\[ \lambda_C = \lambda'_C + \frac{V_{ce}}{R_0} \]
The circuit which models these equations is

\[ i_c' = q_m v_{be} \]

The above circuit is called the hybrid-\( π\) model of the BJT. We can relate \( q_m \) and \( R_π \) as follows:

\[ q_m = \frac{I_c}{V_T} = \frac{\beta I_B}{V_T} = \frac{\beta}{R_π} \]

\[ \Rightarrow \beta = q_m R_π \]

We can relate the currents as follows:
\[ i_c' = q_m V_{be} = q_m i_b R_{\pi} = \beta i_b \]

\[ \Rightarrow i_c' = \beta i_b \]

\[ i_e' = i_b + i_c' = \frac{i_e'}{\beta} + i_c' = \frac{1+\beta}{\beta} i_c' \]

But \[ \alpha = \frac{\beta}{1+\beta} \Rightarrow i_e' = \frac{i_c'}{\alpha} \]

\[ \Rightarrow i_c' = \alpha i_e' \]

Thus we have 3 equations for \( i_c' \):

\[ i_c' = q_m V_{be} = \beta i_b = \alpha i_e' \]

Either of these can be used in analyzing a BJT circuit.

Next, we convert the \( pi \) model to the \( T \) model:

\[ V_{be} = i_b R_{\pi} = \frac{i_e'}{\beta} R_{\pi} = \frac{\alpha i_e'}{\beta} R_{\pi} \]
Let us define the resistance $R_e$ by

$$R_e = \frac{a}{\beta} R_T = \frac{a}{\beta} \frac{V_T}{I_B} = \frac{aV_T}{I_c} = \frac{V_T}{I_E}$$

$$\Rightarrow V_{be} = i_e R_e$$

$$i_c' = \alpha i_e' + \frac{V_{ce}}{R_0}$$

The circuit which models these equations is

![Circuit Diagram]

Where $i_c' = \alpha i_e'$

This is called the T model. In this circuit, $i_c'$ can be calculated from one of the 3 equations

$$i_c' = g_m V_{be} = \beta i_b = \alpha i_e'$$
The Common Emitter Amplifier

The Q point values are calculated with $C_1$, $C_2$, and $C_3$ open circuited. We assume this solution is known.

For a sinusoidal signal, the complex impedance of a capacitor is given by $Z_C = \frac{1}{j\omega C}$. If $\omega C$ is sufficiently large, $|Z_C|$ can be made small enough to be considered an ac short circuit. We assume each $C$ can be considered an ac short circuit.
The ac signal equivalent circuit is as follows:

\[
\begin{align*}
V_i & \quad \| \quad R_{iB} \quad \| \quad R_{IL} \quad \| \quad R_{E} \quad \| \quad R_{out} \quad \| \quad V_o \\
\end{align*}
\]

The input resistance is given by

\[ R_{in} = R_{iB}R_{E}||R_{IL} \]

where \( R_{iB} \) is the small-signal resistance looking into the base. From the \( \pi \) model, this is
\[ R_{iB} = R_{\pi} = \frac{V_T}{I_B}. \]
Note that \( R_{iB} \neq R_{\pi} \) if the emitter is not grounded.

To solve for the output voltage, we make a Thevenin equivalent circuit looking out of the base.
\[ U_{tb} = U_i \frac{R_i R_{11} R_2}{R_i + R_{i11} R_2} \]

\[ R_{tb} = R_{i11} R_{11} R_2 \]

The new circuit is:

To solve for \( V_0 \) and \( R_{out} \), we replace \( Q \) with either the \( \pi \) or the \( T \) model. Let us first use the \( \pi \) model.
\[ V_0 = -\beta i'_c \left( R_{\text{ol}} || R_c || R_L \right) \]

\[ i'_c = \beta i'_b = \beta \frac{V_{tb}}{R_{tb} + R_{\pi}} \]

\[ \Rightarrow V_0 = -\beta \frac{R_{\text{ol}} || R_c || R_L}{R_{tb} + R_{\pi}} V_{tb} = -\beta \frac{R_{\text{ol}} || R_c || R_L}{R_{tb} + R_{\pi}} \frac{R_{\pi}}{R_I + R_{\text{ol}} || R_L} V_{i'_c} \]

Alternate Solution \((g_m = I_c / V_T)\)

\[ i'_c = g_m v_{be} = g_m V_{tb} \frac{R_{\pi}}{R_{tb} + R_{\pi}} \]

But we remember that \(g_m R_{\pi} = \beta \)

\[ \Rightarrow i'_c = \beta \frac{V_{tb}}{R_{tb} + R_{\pi}} \]
This is the same solution obtained above. The output resistance is given by

\[ R_{out} = R_0 \parallel R_C \]

Next, we repeat the solution with the \( T \) model.

\[ V_{tb} = I_b R_{tb} + I_e R_e \]

\[ = \frac{I_e}{1 + \beta} R_{tb} + I_e R_e \]

\[ \Rightarrow I_e = \frac{V_{tb}}{R_{tb} + R_e} \]

\[ V_o = -I_e (R_0 \parallel R_C \parallel R_L) \]
But \( \dot{i}_c = \alpha \dot{i}_e \)

\[
\Rightarrow v_0 = -\alpha \dot{i}_e \left( r_{o11} R_c || R_L \right)
\]

\[
= -\alpha \frac{v_{ib}}{\frac{R_{tb}}{1+\beta} + R_e} \left( r_{o11} R_c || R_L \right)
\]

\[
= -\alpha \frac{r_{o11} R_c || R_L}{\frac{R_{tb}}{1+\beta} + R_e} \frac{R_{11} R_L}{R_{11} + R_{11} R_L} v_i
\]

\( r_{out} = r_{o11} R_c \)

The two solutions are equivalent.

The Common Collector Amplifier
We assume the Q point solution. In the ac signal circuit, make a Thévenin equivalent looking out of the base as in the CE amplifier.

\[ V_{+b} = V_i \frac{R_{11}R_2}{R_i + R_{11}R_2} \]

\[ R_{+b} = R_{11}R_{11}R_2 \]

The pi model circuit is
\[ V_0 = \lambda_e' \left( n_0 \parallel R_E \parallel R_L \right) \]

\[ V_{tb} = \lambda_b' \left( R_{tb} + n \pi \right) + \lambda_e' \left( n_0 \parallel R_E \parallel R_L \right) \]

\[ = \frac{\lambda_e'}{1 + \beta} \left( R_{tb} + n \pi \right) + \lambda_e' \left( n_0 \parallel R_E \parallel R_L \right) \]

\[ \Rightarrow \lambda_e' = \frac{V_{tb}}{\frac{R_{tb} + n \pi}{1 + \beta} + n_0 \parallel R_E \parallel R_L} \]

\[ \Rightarrow V_0 = \frac{n_0 \parallel R_E \parallel R_L}{\frac{R_{tb} + n \pi}{1 + \beta} + n_0 \parallel R_E \parallel R_L} \cdot V_{tb} \]

\[ = \frac{n_0 \parallel R_E \parallel R_L}{\frac{R_{tb} + n \pi}{1 + \beta} + n_0 \parallel R_E \parallel R_L} \cdot \frac{R \parallel R_L}{R_{i} + R \parallel R_L} \cdot V_{\lambda} \]

Note that \[ \frac{V_0}{V_{\lambda}} \leq 1 \]
To solve for $R_{in}$, we must solve for $R_{ie}$.

\[ R_{te} = R_e \parallel R_L \]

Replace $Q$ with the pi model and add a test source at the base.

\[ R_{ie} = \frac{V_t}{I_b} \]
\[ V_t = I_b R \pi + I_e (R_{\text{Ol}} || R_{\text{te}}) \]

\[ = I_b R \pi + (1 + \beta) I_b (R_{\text{Ol}} || R_{\text{te}}) \]

\[ \Rightarrow R_{\text{ib}} = \frac{V_t}{I_b} = R \pi + (1 + \beta) (R_{\text{Ol}} || R_{\text{te}}) \]

The input resistance to the original circuit is

\[ R_{\text{in}} = R_{\text{b}} || R_{\text{el}} || R_{\text{ib}} \]

To solve for \( R_{\text{out}} \), we need the resistance seen looking up into the \( I_e \) branch. We can use the \( T \) model and a test current source to solve for this. Set \( V_i = 0 \Rightarrow V_{ib} = 0 \).
\[ V_t = i_t \cdot R_e + (i_t + i_c') \cdot R_{tb} \]

But \( i_c' = \alpha \cdot i_c' = -\alpha \cdot i_t \)

\[ \Rightarrow V_t = i_t \cdot R_e + i_t \cdot (1-\alpha) \cdot R_{tb} \]

Let \( R_e' = \frac{V_t}{i_t} = R_e + (1-\alpha) \cdot R_{tb} = R_e + \frac{R_{tb}}{1+\beta} \)

The circuit for \( R_{out} \) is:

\[ \text{Rout} = R_e' \parallel R_0 \parallel R_E \]
We assume the Q point solution is known. For the ac analysis, we make a Thévenin equivalent looking out of the emitter.

\[ V_{te} = V_i \frac{R_E}{R_I + R_E} \]

\[ R_{te} = R_E \| R_I \]

In the ac circuit, neither end of \( R_0 \) is grounded, so we omit it to obtain an approximate solution.

Either the pi or the T model can be used. We will use the T model.
\( R_e = \frac{V_T}{I_E} \)

\[ V_o = -i_c' (R_c \parallel R_L) \]

\[ i_c' = \alpha i_e' \]

\[ i_e' = -\frac{V_{te}}{R_{te} + R_e} \]

\[ = + \alpha \frac{R_c \parallel R_L}{R_{te} + R_e} \frac{V_{te}}{R_{te} + R_e} \]

\[ = + \alpha \frac{R_c \parallel R_L}{R_{te} + R_e} \frac{R_E}{R_E + RE} V_i \]

\[ R_{out} = R_c \parallel R_L \]
The resistance seen looking up into the emitter node is $R_e$. Thus the equivalent circuit for $R_{in}$ is

$$R_{in} = R_e \parallel R_E$$

and

$$R_e = \frac{V_T}{I_E}$$
The CE-CC Amplifier

The dc bias solution for $Q_1$ is the same as for the CE amp. The dc bias circuit for $Q_2$ is

$$V_{BB2} = V^+ - Ic_1 R_c1 \quad R_{BB2} = R_c1$$

Thus we can redraw the circuit to obtain
\[ V_{BB2} - V^- = \frac{Ic2}{\beta} R_{BB2} + V_{BE} + \frac{Ic2}{\alpha} R_{E2} \]

\[ \Rightarrow Ic2 = \frac{V_{BB2} - V^- - V_{BE}}{\frac{R_{BB2}}{\beta} + \frac{R_{E2}}{\alpha}} \]

\[ = \frac{V^+ - Ic1 R_{E1} - V^- - V_{BE}}{\frac{R_{BB2}}{\beta} + \frac{R_{E2}}{\alpha}} \]

For \( Q_2 \) to be in the active mode, \( V_{CB2} \) must be positive.

\[ V_{CB2} = Vc2 - V_{B2} \]

\[ = V^+ - (V_{BE} + Ic2 R_{E2} + V^-) \]

\[ = V^+ - (V_{BE} + \frac{Ic2}{\alpha} R_{E2} + V^-) \]
The ac signal circuit is obtained by shorting all capacitors and setting \( V^+ = 0 \) and \( V^- = 0 \). As with the CE amplifier, let

\[
V^+_{\text{th1}} = V_X^+ \frac{R_1 || R_2}{R_1 + R_1 || R_2} \quad R_{\text{th1}} = R_1 || R_1 || R_2
\]

The ac signal circuit is

Note that \( R_{\text{ib2}} \) is the load resistor \( R_{L1} \) for \( Q_1 \). From our common-collector amplifier analysis, it is given by

\[
R_{\text{ib2}} = R_{\text{f2}} + (1 + \beta_2) \left( R_{\text{o2}} || R_{E2} || R_L \right) \frac{1}{R_{\text{te2}}}
\]
Thus we can use our CE amplifier analysis to write

\[ V_{b2} = -\beta \frac{R_o R_e R_{ib2}}{R_{tb1} + R_{\pi 1}} V_{tb1} \]

Because there is no resistor between the \( V_{b2} \) node and the base of \( Q_2 \), it follows that

\[ V_{tb2} = V_{b2} \quad R_{tb2} = 0 \]

Thus we can use our CC amplifier analysis to write

\[ V_o = \frac{R_o R_e R_L}{\frac{R_{\pi}}{1 + \beta} + R_o R_e R_L} \frac{V_{b2}}{V_{tb1}} \]

The overall voltage gain is given by

\[ \frac{V_o}{V_i} = \frac{V_{tb1}}{V_i} \times \frac{V_{b2}}{V_{tb1}} \times \frac{V_o}{V_{b2}} \]

where the 3 gain terms are found from the above analysis.
The Thévenin Equivalent Circuit seen looking into the \( i_e \) branch of the BJT Small-Signal Model.

Consider the ac signal circuit:

\[
\begin{array}{c}
\text{Consider the } \text{ac signal circuit} \\
\end{array}
\]

The \( T \) model is:

\[
\begin{array}{c}
\text{The } T \text{ model is} \\
\end{array}
\]

We wish to solve for the Thévenin equivalent circuit seen looking up into the \( i_e \) branch, the part of the circuit which we need for this.
is as follows

\[ V_{tb} - V_e = \frac{i_e}{1+\beta} R_{tb} + i_e R_e \]

\[ = i_e' \left( \frac{R_{tb}}{1+\beta} + R_e \right) \]

Let us define \( R_e' \) as follows

\[ R_e' = \frac{R_{tb}}{1+\beta} + R_e \]

\[ \Rightarrow V_{tb} - V_e = i_e' R_e' \]

\[ \Rightarrow V_e = V_{tb} - i_e' R_e' \]

The circuit which this equation applies to is
This is the Thévenin equivalent circuit seen looking up into the \( v_e \) branch.

We can now draw a simplified form of the \( T \) model as follows:

Note that the base node \( B \) has been absorbed. We use this model to solve the BJT differential amplifier.