The BJT

BJT Device Equations

Figure 1 shows the circuit symbols for the npn and pnp BJTs. In the active mode, the collector-base junction is reverse biased and the base-emitter junction is forward biased. Because of recombinations of the minority and majority carriers, the equations for the currents can be divided into three regions: low, mid, and high. For the npn device, the currents are given by

Low Level:

\[ i_B = I_{SE} \left[ \exp \left( \frac{v_{BE}}{nV_T} \right) - 1 \right] \]  (1)

\[ i_C = I_S \left[ \exp \left( \frac{v_{BE}}{V_T} \right) - 1 \right] \]  (2)

Mid Level:

\[ i_B = \frac{I_S}{\beta_F} \left[ \exp \left( \frac{v_{BE}}{V_T} \right) - 1 \right] \]  (3)

\[ i_C = I_S \left[ \exp \left( \frac{v_{BE}}{V_T} \right) - 1 \right] \]  (4)

High Level:

\[ i_B = \frac{I_S}{\beta_F} \left[ \exp \left( \frac{v_{BE}}{V_T} \right) - 1 \right] \]  (5)

\[ i_C = I_S \sqrt{\frac{I_K}{I_{SO}}} \left[ \exp \left( \frac{v_{BE}}{2V_T} \right) - 1 \right] \]  (6)

where all leakage currents that are a function of \( v_{CB} \) have been neglected. In the current equations, \( I_S \) is the saturation current and \( \beta_F \) is the mid-level base-to-collector current gain. These are functions of the collector-base voltage and are given by

\[ I_S = I_{SO} \left( 1 + \frac{v_{CB}}{V_A} \right) = I_{SO} \left( 1 + \frac{v_{CE} - v_{BE}}{V_A} \right) \]  (7)

\[ \beta_F = \beta_{FO} \left( 1 + \frac{v_{CB}}{V_A} \right) = \beta_{FO} \left( 1 + \frac{v_{CE} - v_{BE}}{V_A} \right) \]  (8)

Figure 1: BJT circuit symbols.

In the equations for \( i_B \) and \( i_C \), \( V_A \) is the Early voltage and \( I_{SO} \) and \( \beta_{FO} \), respectively, are the zero bias values of \( I_S \) and \( \beta_F \). The constant \( n \) is the emission coefficient or ideality factor of the base-emitter junction. It accounts for recombinations of holes and electrons in the base-emitter
junction at low levels. Its value, typically in the range $1 \leq n \leq 4$, is determined by the slope of
the plot of $\ln(i_C)$ versus $v_{BE}$ at low levels. The default value in SPICE is $n = 1.5$. The constant
$I_{SE}$ is determined by the value of $i_B$ where transition from the low-level to the mid-level region
occurs. The constant $I_K$ is determined by the value of $i_C$ where transition from the mid-level to
the high-level region occurs. Note that $I_S/\beta_F = I_{SO}/\beta_{FO}$ so that $i_B$ is not a function of $v_{CB}$ in the
mid-level region. The equations apply to the pnp device if the subscripts $BE$ and $CB$ are reversed.

Figure 2 shows a typical plot of $i_C$ versus $v_{BE}$ for $v_{CE}$ constant. The plot is called the transfer
characteristics. There is a threshold voltage above which the current appears to increase rapidly.
This voltage is typically 0.5 to 0.6 V. In the forward active region, the base-to-emitter voltage is
typically 0.6 to 0.7 V. Figure 3 shows typical plots of $i_C$ versus $v_{CE}$ for $i_B$ constant. The plots are
called the output characteristics. Note that the slope approaches a constant as $v_{CE}$ is increased. If
the straight line portions of the curves are extended back so that they intersect the $v_{CE}$ axis, they
would intersect at the voltage $v_{CE} = -V_A + v_{BE} \simeq -V_A$. For $v_{CE}$ small, $v_{BE} > v_{CE}$ and the BJT
is in the saturation region.

\[ i_B = I_{SE} \left[ \exp \left( \frac{v_{BE}}{nV_T} \right) - 1 \right] + \frac{I_{SO}}{\beta_{FO}} \left[ \exp \left( \frac{v_{BE}}{V_T} \right) - 1 \right] \] (9)
\[ i_C = \frac{I_S}{K_q} \left[ \exp \left( \frac{v_{BE}}{V_T} \right) - 1 \right] \]  

(10)

where \( K_q \) is given by

\[ K_q = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{I_S}{I_K} \left[ \exp \left( \frac{v_{BE}}{V_T} \right) - 1 \right]} = 1 + \frac{i_C}{I_K} \]  

(11)

Fig. 4 illustrates typical plots of \( \ln (i_C) \) and \( \ln (i_B) \) versus \( v_{BE} \), where it is assumed that \( v_{CB} \) is held constant. At low levels, the \( i_C \) curve exhibits a slope \( m = 1 \) while the \( i_B \) curve exhibits a slope \( m = 1/2 \), where the value \( n = 1.5 \) has been used. At mid levels, both curves exhibit a slope \( m = 1 \). At high levels, the \( i_C \) curve exhibits a slope \( m = 1/2 \) while the \( i_B \) curve exhibits a slope \( m = 1 \). It follows that the ratio of \( i_C \) to \( i_B \) is approximately constant at mid levels and decreases at low and high levels.

![Figure 4: Example plots of \( \ln (i_C) \) and \( \ln (i_B) \) versus \( v_{BE} \).](image)

**Current Gains**

Let the collector and base currents be written as the sum of a dc component and a small-signal ac component as follows:

\[ i_C = I_C + i_c \]  

(12)

\[ i_B = I_B + i_b \]  

(13)

The dc current gain \( \beta_{Fdc} \) is defined as the ratio of \( I_C \) to \( I_B \). It is straightforward to show that it is given by

\[ \beta_{Fdc} = \frac{\beta_F}{1 + \frac{I_C}{I_K} + \frac{\beta_F I_S}{I_C} \left[ \left( 1 + \frac{I_C}{I_S} \right) + \frac{I_C^2}{I_S I_K} \right]^{1/n} - 1} \]  

(14)

Because \( \beta_F \) and \( I_S \) are functions of the collector-base voltage \( V_{CB} \), it follows that \( \beta_{Fdc} \) is a function of both \( I_C \) and \( V_{CB} \). If \( v_{CB} \) is held constant so that the change in \( i_C \) is due to a change in \( v_{BE} \), the small-signal change in base current can be written

\[ i_b = \frac{\partial I_B}{\partial I_C} i_c = \left[ \frac{\partial}{\partial I_C} \left( \frac{I_C}{\beta_{Fdc}} \right) \right] i_c = \frac{i_c}{\beta_{Fdc}} \]  

(15)
where $\beta_{Fac}$ is the small-signal ac current gain given by

$$
\beta_{Fac} = \left[ \frac{\partial}{\partial I_C} \left( \frac{I_C}{\beta_{Fdc}} \right) \right]^{-1} = \frac{\beta_F}{\left( 1 + \frac{2I_C}{I_K} \right) \left[ 1 + \frac{\beta_F I_SE}{nI_S} \left( 1 + \frac{I_C}{I_S} + \frac{I_E^2}{I_S I_K} \right) \right]^{\frac{1}{2}}}
$$

(16)

Note that $\beta_{Fac}$ is defined for a constant $v_{CB}$. In the small-signal models, it is common to define the small-signal ac current gain with $v_{CE}$ constant, i.e. $v_{ce} = 0$. This is defined in the next section, where the symbol $\beta$ is used.

Typical plots of the two current gains as a function of $I_C$ are shown in Fig. 5 where log scales are used. At low levels, the gains decrease with decreasing $I_C$ because the base current decreases at a slower rate than the collector current. At high levels, the gains decrease with increasing $I_C$ because the collector current increases at a slower rate than the base current. At mid levels, both gains are approximately constant and have the same value. In the figure, the mid-level range is approximately two decades wide.

![Current Gains](image)

Figure 5: Log-log plots of $\beta_{Fdc}$ and $\beta_{Fac}$ as functions of $I_C$.

The emitter-collector dc current gain $\alpha_{Fdc}$ is defined as the ratio of the dc collector current $I_C$ to the dc emitter current $I_E$. To solve for this, we can write

$$I_E = I_B + I_C = \left( \frac{1}{\beta_{Fdc}} + 1 \right) I_C = \frac{1 + \beta_{Fdc} I_C}{\beta_{Fdc}}$$

(17)

It follows that

$$\alpha_{Fdc} = \frac{I_C}{I_E} = \frac{\beta_{Fdc}}{1 + \beta_{Fdc}}$$

(18)

Thus the dc currents are related by the equations

$$I_C = \beta_{Fdc} I_B = \alpha_{Fdc} I_E$$

(19)

**Bias Equation**

Figure 6(a) shows the BJT with the external circuits represented by Thévenin dc circuits. If the BJT is biased in the active region, we can write

$$V_{BB} - V_{EE} = I_B R_{BB} + V_{BE} + I_E R_{EE}$$

$$= \frac{I_C}{\beta_{Fdc}} R_{BB} + V_{BE} + \frac{I_C}{\alpha_{Fdc}} R_{EE}$$

(20)
This equation can be solved for $I_C$ to obtain

$$ I_C = \frac{V_{BB} - V_{EE} - V_{BE}}{R_{BB}/\beta_{Fdc} + R_{EE}/\alpha_{Fdc}} \quad (21) $$

It can be seen from Fig. 2 that large changes in $I_C$ are associated with small changes in $V_{BE}$. This makes it possible to calculate $I_C$ by assuming typical values of $V_{BE}$. Values in the range from 0.6 to 0.7 V are commonly used. In addition, $\beta_{Fdc}$ and $\alpha_{Fdc}$ are functions of $I_C$ and $V_{CB}$. Mid-level values are commonly assumed for the current gains. Typical values are $\beta_{Fdc} = 100$ and $\alpha_{Fdc} = 1/1.01$.

![Diagram](image)

**Example 1** Figure 6(b) shows a BJT dc bias circuit. It is given that $V^+ = 15$ V, $R_1 = 20 \, k\Omega$, $R_2 = 10 \, k\Omega$, $R_3 = R_4 = 3 \, k\Omega$, $R_5 = R_6 = 2 \, k\Omega$. Solve for $I_{C1}$ and $I_{C2}$. Assume $V_{BE} = 0.7$ V and $\beta_{Fdc} = 100$ for each transistor.

**Solution.** For $Q_1$, we have $V_{BB1} = V^+ R_2/(R_1 + R_2)$, $R_{BB1} = R_1 \parallel R_2$, $V_{EE1} = -I_{B2} R_4 = -I_{C2} R_4/\beta_{Fdc}$, $V_{EE1} = 0$, and $R_{EE1} = R_4$. For $Q_2$, we have $V_{BB2} = I_{E1} R_4 = I_{C1} R_4/\alpha_{Fdc}$, $R_{BB2} = R_4$, $V_{EE2} = 0$, $R_{EE2} = R_6$. Thus the bias equations are

$$ V^+ \frac{R_2}{R_1 + R_2} + \frac{I_{C2}}{\beta_{Fdc}} R_4 = V_{BE} + \frac{I_{C1}}{\beta_{Fdc}} R_1 \parallel R_2 + \frac{I_{C1}}{\alpha_{Fdc}} R_4 $$

$$ \frac{I_{C1}}{\alpha_{Fdc}} R_4 = V_{BE} + \frac{I_{C2}}{\beta_{Fdc}} R_4 + \frac{I_{C2}}{\alpha_{Fdc}} R_6 $$

These equations can be solved simultaneously to obtain $I_{C1} = 1.41$ mA and $I_{C2} = 1.74$ mA.

**Small-Signal Models**

There are two small-signal circuit models which are commonly used to analyze BJT circuits. These are the hybrid-$\pi$ model and the $T$ model. The two models are equivalent and give identical results. They are described below.

**Hybrid-$\pi$ Model**

Let each current and voltage be written as the sum of a dc component and a small-signal ac component. The currents are given by Eqs. (12) and (13). The voltages can be written

$$ v_{BE} = V_{BE} + v_{be} \quad (22) $$
\[ v_{CB} = V_{CB} + v_{cb} \]  

If the ac components are sufficiently small, \( i_c \) can be written
\[
\begin{align*}
i_c &= \frac{\partial I_C}{\partial V_{BE}} v_{be} + \frac{\partial I_C}{\partial V_{CB}} v_{cb} = \frac{\partial I_C}{\partial V_{BE}} v_{be} + \frac{\partial I_C}{\partial V_{CB}} (v_{ce} - v_{be}) \\
&= \left( \frac{\partial I_C}{\partial V_{BE}} - \frac{\partial I_C}{\partial V_{CB}} \right) v_{be} + \frac{\partial I_C}{\partial V_{CB}} v_{ce} = g_m v_{be} + \frac{v_{ce} - v_{be}}{r_0}
\end{align*}
\]

This equation defines the small-signal transconductance \( g_m \) and the collector-emitter resistance \( r_0 \). From Eqs. (8) and (10), it follows that \( r_0 \) is given by
\[
r_0 = \left( \frac{\partial I_C}{\partial V_{CB}} \right)^{-1} = \left\{ \frac{K_t I_s}{V_A} \exp \left( \frac{v_{BE}}{V_T} \right) - 1 \right\}^{-1}
= \frac{V_A + V_{CB}}{I_C}
\]

To solve for \( g_m \), we first solve for \( \frac{\partial I_C}{\partial V_{BE}} \). Eqs. (10) and (11) can be combined to write
\[
I_C + \frac{I_C^2}{I_K} = I_S \left[ \exp \left( \frac{V_{BE}}{V_T} \right) - 1 \right]
\]

It follows from this equation that \( \frac{\partial I_C}{\partial V_{BE}} \) is given by
\[
\frac{\partial I_C}{\partial V_{BE}} = \frac{I_S \exp \left( \frac{V_{BE}}{V_T} \right)}{V_T \left( 1 + 2 I_C / I_K \right)} = \frac{I_C \left( 1 + I_C / I_K \right) + I_S}{V_T \left( 1 + 2 I_C / I_K \right)}
\]

The transconductance is given by
\[
g_m = \frac{\partial I_C}{\partial V_{BE}} - \frac{\partial I_C}{\partial V_{CB}} = \frac{I_C \left( 1 + I_C / I_K \right) + I_S}{V_T \left( 1 + 2 I_C / I_K \right)} - \frac{1}{r_0}
\]

It is clear from Eq. (9) that \( i_B \) is a function of \( v_{BE} \) only. We wish to solve for the small-signal ac base current given by \( i_b = (\partial I_B / \partial V_{BE}) v_{be} \). This equation defines the small-signal ac base-emitter resistance \( r_\pi = v_{be} / i_b = (\partial I_B / \partial V_{BE})^{-1} \). Although Eq. (9) can be used to solve for this, we use a different approach. The small-signal ac collector current can be written
\[
i_c = g_m v_{be} + \frac{v_{ce} - v_{be}}{r_0} = g_m v_{be} + v_{ce} \frac{1}{r_0} + \frac{v_{ce} - v_{be}}{r_0}
\]

It follows from this equation that
\[
\left( g_m + \frac{1}{r_0} \right) v_{be} = \beta_{Fac} i_b
\]

Thus \( r_\pi \) is given by
\[
r_\pi = \frac{v_{be}}{i_b} = \frac{\beta_{Fac}}{g_m + 1 / r_0}
\]

The small-signal ac current gain \( \beta \) is defined as the ratio of \( i_c \) to \( i_b \) with \( v_{CE} \) constant, i.e. \( v_{ce} = 0 \). To solve for this, we can write for \( i_c \)
\[
i_c = g_m v_{be} + \frac{v_{ce} - v_{be}}{r_0} = g_m i_b r_\pi + \frac{v_{ce} - v_{be}}{r_0} = \beta i_b + \frac{v_{ce} - v_{be}}{r_0}
\]
It follows from this that $\beta$ is given by

$$\beta = g_m r_\pi = \frac{g_m \beta_F \pi}{g_m + 1/r_0} \quad (33)$$

Thus far, we have neglected the base spreading resistance $r_x$. This is the ohmic resistance of the base contact in the BJT. When it is included in the model, it appears in series with the base lead. Because the base region is very narrow, the connection exhibits a resistance which often cannot be neglected. Fig. 7(a) shows the hybrid-π small-signal model with $r_\pi$ included. The currents are given by

$$i_c = i'_c + \frac{v_{ce}}{r_0} \quad (34)$$
$$i'_c = g_m v_\pi = \beta i_b \quad (35)$$
$$i_b = \frac{v_{be}}{r_\pi} \quad (36)$$

where $r_0$, $g_m$, $\beta$, and $r_\pi$ are given above.

![Figure 7: (a) Hybrid-π model. (b) T model.](image)

The equations derived above are based on the Gummel-Poon model of the BJT in the forward active region. The equations are often approximated by assuming that the mid-level current equations hold. In this case, $\beta_{Fdc}$, $\beta_{Fac}$, $r_0$, $g_m$, $r_\pi$, and $\beta$ are given by

$$\beta_{Fdc} = \beta_{Fac} = \beta_F \quad (37)$$
$$r_0 = \frac{V_{CB} + V_A}{I_C} \quad (38)$$
$$g_m = \frac{I_C + I_S}{V_T} - \frac{1}{r_0} \approx \frac{I_C}{V_T} \quad (39)$$
$$r_\pi = \frac{\beta_F \pi}{g_m + 1/r_0} \approx \frac{V_T}{I_B} \quad (40)$$
$$\beta = g_m r_\pi \approx \frac{I_C}{I_B} = \beta_{Fdc} \quad (41)$$

The three approximations in these equations are commonly used for hand calculations.
T Model

The T model replaces the resistor $r_\pi$ in series with the base with a resistor $r_e$ in series with the emitter. This resistor is called the emitter intrinsic resistance. To solve for $r_e$, we first solve for the small-signal ac emitter-to-collector current gain $\alpha$. In Fig. 7, the current $i'_e$ can be written

$$i'_e = i_b + i'_c = \left(\frac{1}{\beta} + 1\right) i'_c = \frac{1 + \beta}{\beta} i'_c = \frac{i'_c}{\alpha}$$

(42)

where $\alpha$ is given by

$$\alpha = \frac{i'_c}{i'_e} = \frac{\beta}{1 + \beta}$$

(43)

Thus the current $i'_c$ can be written

$$i'_c = \alpha i'_e$$

(44)

The voltage $v_\pi$ can be related to $i'_e$ as follows:

$$v_\pi = i_b r_\pi = \frac{\alpha i'_c}{\beta} r_\pi = i'_e \frac{r_\pi}{1 + \beta}$$

(45)

It follows that the intrinsic emitter resistance is given by

$$r_e = \frac{v_\pi}{i'_e} = \frac{r_\pi}{1 + \beta} \simeq \frac{V_T}{(1 + \beta_{F_{dc}}) I_B} = \frac{V_T}{I_E}$$

(46)

where the approximation is based on Eqs. (40) and (41). It is often used for hand calculations.

The T model of the BJT is shown in Fig. 7(b). The currents in both the $\pi$ and T models are related by the equations

$$i'_c = g_m v_\pi = \beta i_b = \alpha i'_e$$

(47)

Small-Signal Equivalent Circuits

Several equivalent circuits are derived below which facilitate writing small-signal low-frequency equations for the BJT. We assume that the circuits external to the device can be represented by Thévenin equivalent circuits. The Norton equivalent circuit seen looking into the collector and the Thévenin equivalent circuits seen looking into the base and the emitter are derived. Although the T model is used for the derivation, identical results are obtained with the hybrid-$\pi$ model. Several examples are given which illustrate use of the equivalent circuits.

Simplified T Model

Figure 8 shows the T model with a Thévenin source in series with the base. We wish to solve for an equivalent circuit in which the source $i'_e$ connects from the collector node to ground rather than from the collector node to the $B'$ node. The first step is to replace the source $i'_e$ with two identical series sources with the common node grounded. The circuit is shown in Fig. 9(a).

For the circuit in Fig. 9(a), we can write

$$v_e = v_{tb} - \frac{i'_e}{1 + \beta} (R_{tb} + r_x) - i'_e r_e = \hat{v}_{tb} - i'_e \left(\frac{R_{tb} + r_x}{1 + \beta} + r_e\right)$$

(48)

Let us define the resistance $r'_e$ by

$$r'_e = \frac{R_{tb} + r_x + r_e}{1 + \beta} = \frac{R_{tb} + r_x + r_\pi}{1 + \beta}$$

(49)
With this definition, $v_e$ is given by

$$v_e = v_{tb} - i'_e r'_e$$

The circuit which models this equation is shown in Fig. 9(b). This will be called the simplified T model. It predicts the same emitter and collector currents as the circuit in Fig. 8. Note that the resistors $R_{tb}$ and $r_x$ do not appear in this circuit. They are part of the resistor $r'_e$.

**Norton Collector Circuit**

The Norton equivalent circuit seen looking into the collector can be used to solve for the response of the common-emitter and common-base stages. It consists of a parallel current source $i_{c(sc)}$ and resistor $r_{ic}$ from the collector to signal ground. Fig. 10(a) shows the BJT with Thévenin sources connected to its base and emitter. With the collector grounded, the collector current is the short-circuit or Norton collector current. To solve for this, we use the simplified T model in Fig. 10(b). We use superposition of $v_{tb}$ and $v_{te}$ to solve for $i_{c(sc)}$.

With $v_{te} = 0$, it follows from Fig. 10(b) that

$$i_{c(sc)} = \alpha i'_e + i_0 = \alpha i'_e - i'_e \frac{R_{te}}{r_0 + R_{te}}$$

$$= \frac{v_{tb}}{r'_e + R_{te} r_0} \left( \alpha - \frac{R_{te}}{r_0 + R_{te}} \right)$$

$$= \frac{1}{1 + \beta} \left( \frac{v_{tb}}{r'_e + R_{te} r_0} \left( \alpha - \frac{R_{te}}{r_0 + R_{te}} \right) \right)$$

$$= \frac{v_{tb}}{r'_e + (1 + \beta) R_{te} r_0} \left( \alpha - \frac{R_{te}}{r_0 + R_{te}} \right)$$
With $v_{tb} = 0$, we have

$$i_{c(sc)} = \alpha i'_e + i_0 = \alpha i_e + i_0 = -\frac{r_0}{r_0 + r'_e} + i_e \frac{r'_e}{r_0 + r'_e}$$

$$= -\frac{v_{te}}{R_{te} + r'_e} \left[ \alpha r_0 + r'_e \right] \frac{r_0}{r_0 + r'_e}$$

$$= -\frac{v_{te}}{R_{te} + r'_e} \left[ \alpha r_0 + r'_e \right] \frac{r_0}{r_0 + r'_e}$$

These equations can be combined to obtain

$$i_{c(sc)} = \frac{v_{tb}}{r'_e + R_{te} + r_0} \left( \alpha - \frac{R_{te}}{r_0 + r'_e} \right) - \frac{v_{te}}{R_{te} + r'_e} \left[ \alpha r_0 + r'_e \right] \frac{r_0}{r_0 + r'_e}$$

This equation is of the form

$$i_{c(sc)} = G_{mb} v_{tb} - G_{me} v_{te}$$

where

$$G_{mb} = \frac{1}{r'_e + R_{te} + r_0} \left( \alpha - \frac{R_{te}}{r_0 + r'_e} \right) = \frac{\alpha}{r'_e + R_{te} + r_0} \frac{r_0 - R_{te}/\beta}{r_0 + R_{te}}$$

$$G_{me} = \frac{1}{r'_e + R_{te} + r_0} \left[ \alpha r_0 + r'_e \right] = \frac{\alpha}{r'_e + R_{te} + r_0} \frac{r_0 + r'_e/\alpha}{r_0 + R_{te}}$$

The next step is to solve for the resistance seen looking into the collector with $v_{tb} = v_{te} = 0$. Figure 11(a) shows the simplified T model with a test source connected to the collector. The resistance seen looking into the collector is given by $r_{ic} = v_{t}/i_{c}$. To solve for $r_{ic}$, we can write

$$i_{c} = \alpha i'_e + i_0 = -\alpha i_0 \frac{R_{te}}{r'_e + R_{te}} + i_0$$

$$= \frac{v_{t}}{r_0 + r'_e} \left( 1 - \alpha \frac{R_{te}}{r'_e + R_{te}} \right)$$

It follows that $r_{ic}$ is given by

$$r_{ic} = \frac{v_{t}}{i_{c}} = \frac{r_0 + r'_e}{1 - \alpha R_{te}/(r'_e + R_{te})}$$

The Norton equivalent circuit seen looking into the collector is shown in Fig. 11(b).
Figure 11: (a) Circuit for calculating $r_{ic}$. (b) Norton collector circuit.

For the case $r_0 \gg R_{te}$ and $r_0 \gg r'_e$, we can write

$$i_{c(sc)} = G_m (v_{tb} - v_{te})$$

(59)

where

$$G_m = \frac{\alpha}{r'_e + R_{te}}$$

(60)

The value of $i_{c(sc)}$ calculated with this approximation is simply the value of $\alpha i'_e$, where $i'_e$ is calculated with $r_0$ considered to be an open circuit. The term “$r_0$ approximations” is used in the following when $r_0$ is neglected in calculating $i_{c(sc)}$ but not neglected in calculating $r_{ic}$.

Thévenin Emitter Circuit

The Thévenin equivalent circuit seen looking into the emitter is useful in calculating the response of common-collector stages. It consists of a voltage source $v_{e(oc)}$ in series with a resistor $r_{ie}$ from the emitter node to signal ground. Fig. 12(a) shows the BJT symbol with a Thévenin source connected to the base. The resistor $R_{tc}$ represents the external load resistance in series with the collector. With the emitter open circuited, we denote the emitter voltage by $v_{e(oc)}$. The voltage source in the Thévenin emitter circuit has this value. To solve for it, we use the simplified T model in Fig. 12(b).

Figure 12: (a) BJT with Thévenin source connected to the base. (b) Simplified T model circuit for calculating $v_{e(oc)}$.

The current $i'_e$ can be solved for by superposition of the sources $v_{tb}$ and $\alpha i'_e$. It is given by

$$i'_e = \frac{v_{tb}}{r'_e + r_0 + R_{tc}} + \alpha \frac{R_{tc}}{r'_e + r_0 + R_{tc}}$$

(61)
This can be solved for \( i'_e \) to obtain

\[
i'_e = \frac{v_{tb}}{r'_e + r_0 + (1 - \alpha) R_{tc}} = \frac{v_{tb}}{r'_e + r_0 + R_{tc} / (1 + \beta)}
\]

(62)

The open-circuit emitter voltage is given by

\[
v_{e(oc)} = v_{tb} - \frac{i'_e r'_e}{v_{tb} r'_e + r_0 + R_{tc} / (1 + \beta)}
\]

(63)

We next solve for the resistance seen looking into the emitter node. It can be solved for as the ratio of the open-circuit emitter voltage \( v_{e(oc)} \) to the short-circuit emitter current. The circuit for calculating the short-circuit current is shown in Fig. 13(a). By superposition of \( i'_e \) and \( \alpha i'_e \), we can write

\[
i_{e(sc)} = i'_e - \alpha i'_e \frac{R_{tc}}{r_0 + R_{tc}} = \frac{i'_e r_0 + (1 - \alpha) R_{tc}}{r_0 + R_{tc}}
\]

(64)

The resistance seen looking into the emitter is given by

\[
r_{ie} = \frac{v_{e(oc)}}{i_{e(sc)}} = \frac{r_0 + R_{tc}}{r'_e r_0 + r_0 + R_{tc} / (1 + \beta)}
\]

(65)

The Thévenin equivalent circuit seen looking into the emitter is shown in Fig. 13(b).

![Figure 13: (a) Circuit for calculating \( i_{e(sc)} \). (b) Thévenin emitter circuit.](image)

**Thévenin Base Circuit**

Although the base is not an output terminal, the Thévenin equivalent circuit seen looking into the base is useful in calculating the base current. It consists of a voltage source \( v_{b(oc)} \) in series with a resistor \( r_{ib} \) from the base node to signal ground. Fig. 14(a) shows the BJT symbol with a Thévenin source connected to its emitter. Fig. 14(b) shows the T model for calculating the open-circuit base voltage. Because \( i_b = 0 \), it follows that \( i'_e = 0 \). Thus there is no drop across \( r_x \) and \( r_e \) so that \( v_{b(oc)} \) is given by

\[
v_{b(oc)} = v_e = v_{te} \frac{r_0 + R_{tc}}{R_{tc} + r_0 + R_{tc}}
\]

(66)

The next step is to solve for the resistance seen looking into the base. It can be calculated by setting \( v_{te} = 0 \) and connecting a test current source \( i_t \) to the base. It is given by \( r_{ib} = v_b / i_t \). Fig. 15(a) shows the T circuit for calculating \( v_b \), where the current source \( \beta i_t \) has been divided
into identical series sources with their common node grounded to simplify use of superposition. By
superposition of $i_t$ and the two $\beta i_t$ sources, we can write

$$v_b = i_t r_x + (i_t + \beta i_t) [r_e + (r_0 + R_{te})] - \beta i_t \frac{R_{tc} R_{te}}{r_0 + R_{te} + r_0}$$ (67)

This can be solved for $r_{ib}$ to obtain

$$r_{ib} = \frac{v_b}{i_t} = r_x + (1 + \beta) [r_e + (r_0 + R_{te})] - \frac{\beta R_{tc} R_{te}}{R_{te} + r_0 + R_{te}}$$ (68)

The Thévenin base circuit is shown in Fig. 15(b).

Figure 15: (a) Circuit for calculating $v_b$. (b) Thévenin base circuit.

Summary of Models

Figure 16 summarizes the four equivalent circuits derived above.

Example Amplifier Circuits

This section describes several examples which illustrate the use of the small-signal equivalent circuits
derived above to write by inspection the voltage gain, the input resistance, and the output resistance of both single-stage and two-stage amplifiers.
The Common-Emitter Amplifier

Figure 17(a) shows the ac signal circuit of a common-emitter amplifier. We assume that the bias solution and the small-signal resistances $r_e$ and $r_0$ are known. The output voltage and output resistance can be calculated by replacing the circuit seen looking into the collector by the Norton equivalent circuit of Fig. 11(b). With the aid of this circuit, we can write

\[ v_o = -i_{c(sc)} (r_{ic}) R_{tc} v_{tb} \]

\[ r_{out} = r_{ic} R_{tc} \]

where $G_{mb}$ and $r_{ic}$, respectively, are given by Eqs. (55) and (58). The input resistance is given by

\[ r_{in} = R_{tb} + r_{ib} \]

where $r_{ib}$ is given by Eq. (68).

Figure 17: (a) Common-emitter amplifier. (b) Common-collector amplifier. (c) Common-base amplifier.
The Common-Collector Amplifier

Figure 17(b) shows the ac signal circuit of a common-collector amplifier. We assume that the bias solution and the small-signal resistances $r'_e$ and $r_0$ are known. The output voltage and output resistance can be calculated by replacing the circuit seen looking into the emitter by the Thévenin equivalent circuit of Fig. 13(b). With the aid of this circuit, we can write

$$v_o = v_{e(oc)} \frac{R_{te}}{r_{ie} + R_{te}} = \frac{r_0 + R_{te}/(1 + \beta)}{r'_e + r_0 + R_{te}/(1 + \beta)} \frac{R_{te}}{r_{ie} + R_{te}} v_{lb}$$  \hspace{1cm} (72)$$

$$r_{out} = r_{ie}||R_{te}$$  \hspace{1cm} (73)$$

where $r_{ie}$ is given by Eq. (65). The input resistance is given by

$$r_{in} = R_{tb} + r_{ib}$$  \hspace{1cm} (74)$$

where $r_{ib}$ is given by Eq. (68).

The Common-Base Amplifier

Figure 17(c) shows the ac signal circuit of a common-base amplifier. We assume that the bias solution and the small-signal parameters $r'_e$ and $r_0$ are known. The output voltage and output resistance can be calculated by replacing the circuit seen looking into the collector by the Norton equivalent circuit of Fig. 11(b). The input resistance can be calculated by replacing the circuit seen looking into the emitter by the Thévenin equivalent circuit of Fig. 13 with $v_{e(oc)} = 0$. With the aid of these circuits, we can write

$$v_o = -i_{c1(oc)} (r_{ic}||R_{tc}) = G_{me} (r_{ic}||R_{tc}) v_{te}$$  \hspace{1cm} (75)$$

$$r_{out} = r_{ie}||R_{te}$$  \hspace{1cm} (76)$$

$$r_{in} = R_{tb} + r_{ie}$$  \hspace{1cm} (77)$$

where $G_{me}$, $r_{ic}$, and $r_{ie}$, respectively, are given by Eqs. (56), (58), and (65).

The CE/CC Amplifier

Figure 18(a) shows the ac signal circuit of a two-stage amplifier consisting of a CE stage followed by a CC stage. Such a circuit is used to obtain a high voltage gain and a low output resistance. The voltage gain can be written

$$\frac{v_o}{v_{lb1}} = \frac{i_{c1(oc)}}{v_{lb1}} \times \frac{v_{lb2}}{i_{c1(oc)}} \times \frac{v_{e2(oc)}}{v_{lb2}} \times \frac{v_o}{v_{e2(oc)}}$$

$$= G_{mb1} \left[ - (r_{ic1}||R_{C1}) \right] \frac{r_0}{r'_e + r_0 + r_{ie2} + R_{te2}}$$  \hspace{1cm} (78)$$

where $r'_e$ is calculated with $R_{lb2} = r_{ic1}||R_{C1}$. The input and output resistances are given by

$$r_{in} = R_{tb1} + r_{ib1}$$  \hspace{1cm} (79)$$

$$r_{out} = r_{ie2}||R_{te2}$$  \hspace{1cm} (80)$$

Although not a part of the solution, the resistance seen looking out of the collector of $Q_1$ is $R_{tc1} = R_{C1}||r_{ib2}$.  

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Figure 18: (a) CE-CC amplifier. (b) Cascode amplifier.

The Cascode Amplifier

Figure 18(b) shows the ac signal circuit of a cascode amplifier. The voltage gain can be written

\[
\frac{v_o}{v_{tb1}} = \frac{i_c1(sc)}{v_{tb1}} \times \frac{v_{te2}}{i_c1(sc)} \times \frac{i_c2(sc)}{v_{te2}} \times \frac{i_c2(sc)}{v_o} = G_m1 \left( -r_{ic1} \right) \left( -G_{me2} \right) \left( -r_{ic2} \right) R_{tc2}
\]

where \( G_{me2} \) and \( r_{ic2} \) are calculated with \( R_{te2} = r_{ic1} \). The input and output resistances are given by

- \( r_{in} = R_{tb1} + r_{ib1} \)
- \( r_{out} = R_{tc2} \left/ r_{ic2} \right. \)

The resistance seen looking out of the collector of \( Q_1 \) is \( R_{tc1} = r_{ie2} \).

A second cascode amplifier is shown in Fig. 19(a) where a pnp transistor is used for the second stage. The voltage gain is given by

\[
\frac{v_o}{v_{tb1}} = \frac{i_c1(sc)}{v_{tb1}} \times \frac{v_{te2}}{i_c1(sc)} \times \frac{i_c2(sc)}{v_{te2}} \times \frac{i_c2(sc)}{v_o} = G_m1 \left( -r_{ic1} \right) R_{C1} \left( -G_{me2} \right) \left( -r_{ic2} \right) R_{tc2}
\]

The expressions for \( r_{in} \) and \( r_{out} \) are the same as for the cascode amplifier in Fig. 18(b). The resistance seen looking out of the collector of \( Q_1 \) is \( R_{tc1} = R_{C1} \left/ r_{ie2} \right. \).

The Differential Amplifier

Figure 19(b) shows the ac signal circuit of a differential amplifier. For the case of an active tail bias supply, the resistor \( R_Q \) represents its small-signal ac resistance. We assume that the transistors are identical, biased at the same currents and voltages, and have identical small-signal parameters. Looking out of the emitter of \( Q_1 \), the Thévenin voltage and resistance are given by

\[
v_{te1} = v_{e2(oc)} \frac{R_Q}{R_Q + R_E + r_{ie}} = v_{tb2} \frac{r_0 + R_{te}/(1 + \beta)}{r_e + r_0 + R_{te}/(1 + \beta)} \frac{R_Q}{R_Q + R_E + r_{ie}}
\]
where \( v \) by symmetry, the input resistance seen by both stages in which the signal resistance seen looking into either output is \( R \). Thus the \( v \) solves for \( q \) to change by the same amounts. If \( 2 \), then the \( v \) with \( \frac{v}{2} \) to be grounded and the circuit can be divided into two common-emitter nodes can be obtained by replacing \( v_{tb1} \) and \( v_{qb2} \) with differential and common-mode components as follows:

\[
v_{tb1} = v_{i(cm)} + \frac{v_{i(d)}}{2}
\]

\[
v_{tb2} = v_{i(cm)} - \frac{v_{i(d)}}{2}
\]

where \( v_{i(d)} = v_{tb1} - v_{tb2} \) and \( v_{i(cm)} = (v_{tb1} + v_{tb2}) / 2 \). Superposition of \( v_{i(d)} \) and \( v_{i(cm)} \) can be used to solve for \( v_{o1} \) and \( v_{o2} \). With \( v_{i(cm)} = 0 \), the effects of \( v_{tb1} = v_{i(d)}/2 \) and \( v_{tb2} = -v_{i(d)}/2 \) are to cause \( v = 0 \). Thus the \( v \) node can be grounded and the circuit can be divided into two common-emitter stages in which \( R_{te(d)} = R_E \) for each transistor. In this case, \( v_{o1(d)} \) can be written

\[
v_{o1(d)} = \frac{i_{c1(s)}}{v_{tb1(d)}} \times v_{o1(d)} - v_{o2(d)} = G_{m(d)} \left( -r_{ic(d)} || R_{te} \right) \left( \frac{v_{i(d)}}{2} \right)
\]

By symmetry \( v_{o2(d)} = -v_{o1(d)} \).

With \( v_{i(d)} = 0 \), the effects of \( v_{tb1} = v_{tb2} = v_{i(cm)} \) are to cause the emitter currents in \( Q_1 \) and \( Q_2 \) to change by the same amounts. If \( R_Q \) is replaced by two parallel resistors of value \( 2R_Q \), it

**Figure 19:** (a) Second cascode amplifier. (b) Differential amplifier.

\[
R_{te1} = R_E + R_Q || (R_E + r_{ie})
\]

The small-signal collector voltage of \( Q_1 \) is given by

\[
v_{o1} = -i_{c1(s)} (r_{ie} || R_{tc}) = -(G_{mb} v_{tb1} - G_{me} v_{o1}) (r_{ie} || R_{tc})
\]

\[
= G_{mb} (r_{ie} || R_{tc}) v_{tb1} + G_{me} \frac{r_0 + r_{tc} / (1 + \beta)}{r_t^e + r_0 + r_{tc} / (1 + \beta)} R_Q v_{tb2}
\]

By symmetry, \( v_{o2} \) is obtained by interchanging the subscripts 1 and 2 in this equation. The small-signal resistance seen looking into either output is

\[
r_{out} = R_{tc} || r_{ie}
\]

where \( r_{ie} \) calculated from Eq. (58) with \( R_{te} = R_E + R_Q || (R_E + r_{ie}) \). Although not labeled on the circuit, the input resistance seen by both \( v_{tb1} \) and \( v_{tb2} \) is \( r_{in} = r_{ib} \).

A second solution of the diff amp can be obtained by replacing \( v_{tb1} \) and \( v_{tb2} \) with differential and common-mode components as follows:

\[
v_{tb1} = v_{i(cm)} + \frac{v_{i(d)}}{2}
\]

\[
v_{tb2} = v_{i(cm)} - \frac{v_{i(d)}}{2}
\]
follows by symmetry that the circuit can be separated into two common-emitter stages each with \( R_{te(cm)} = R_E + 2R_Q \). In this case, \( v_{o1(cm)} \) can be written

\[
v_{o1(cm)} = \frac{i_{c1(sc)}}{v_{bb1(cm)}} v_{ib1(cm)} = G_{m(cm)} \left(-r_{ie(cm)}\|R_{tc}\right) v_i(cm)
\]

\[
= G_{m(cm)} \left(-r_{ie(cm)}\|R_{tc}\right) \frac{v_{bb1} + v_{ib2}}{2}
\]

(88)

By symmetry \( v_{o2(cm)} = v_{o1(cm)} \).

Because \( R_{te} \) is different for the differential and common-mode circuits, \( G_m \), \( r_{ie} \), and \( r_{ib} \) are different. However, the total solution \( v_{o1} = v_{o1(d)} + v_{o1(cm)} \) is the same as that given by Eq. (83), and similarly for \( v_{o2} \). The small-signal base currents can be written \( i_{b1} = v_i(cm)/r_{ib(cm)} + v_i(d)/r_{ib(d)} \) and \( i_{b2} = v_i(cm)/r_{ib(cm)} - v_i(d)/r_{ib(d)} \). If \( R_Q \to \infty \), the common-mode solutions are zero. In this case, the differential solutions can be used for the total solutions. If \( R_Q \gg R_E + r_{ie} \), the common-mode solutions are often approximated by zero.

**Small-Signal High-Frequency Models**

Figure 20 shows the hybrid-\( \pi \) and T models for the BJT with the base-emitter capacitance \( c_\pi \) and the base-collector capacitance \( c_\mu \) added. The capacitor \( c_{cs} \) is the collector-substrate capacitance which is present in monolithic integrated-circuit devices but is omitted in discrete devices. These capacitors model charge storage in the device which affects its high-frequency performance. The capacitors are given by

\[
c_\pi = c_{je} + \frac{\tau_F I_C}{V_T}
\]

(89)

\[
c_\mu = \frac{c_{jc}}{[1 + V_{CB}/\phi_C]^m_c}
\]

(90)

\[
c_{cs} = \frac{c_{jcs}}{[1 + V_{CS}/\phi_C]^m_c}
\]

(91)

where \( I_C \) is the dc collector current, \( V_{CB} \) is the dc collector-base voltage, \( V_{CS} \) is the dc collector-substrate voltage, \( c_{je} \) is the zero-bias junction capacitance of the base-emitter junction, \( \tau_F \) is the forward transit time of the base-emitter junction, \( c_{jc} \) is the zero-bias junction capacitance of the base-collector junction, \( c_{jcs} \) is the zero-bias collector-substrate capacitance, \( \phi_C \) is the built-in potential, and \( m_c \) is the junction exponential factor. For integrated circuit lateral pnp transistors, \( c_{cs} \) is replaced with a capacitor \( c_{bs} \) from base to substrate, i.e. from the B node to ground.

Figure 20: High-frequency small-signal models of the BJT. (a) Hybrid-\( \pi \) model. (b) T model.
In these models, the currents are related by

\[ i'_c = g_m v_\pi = \beta i'_b = \alpha i'_e \]  

(92)

These relations are the same as those in Eq. (47) with \( i_b \) replaced with \( i'_b \).