ECE 6416 Practice Problems 1

1. The probability density function for a zero-mean random noise voltage is

$$p(v) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{v^2}{2\sigma^2}\right)$$

where σ is the standard deviation which corresponds to the rms value of the voltage. What is the probability density function for the voltage if a dc offset voltage V_{DC} is added to the noise?

2. This problem illustrates how the mean-square value of the sum of two functions is calculated. The time functions are periodic sinusoids. Some useful trigonometric relations are

$$\cos x \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$

$$\langle \cos \omega t \rangle = 0 \qquad \left\langle \cos^2 \omega t \right\rangle = \frac{1}{2}$$

$$\langle \cos \omega_1 t \cos \omega_2 t \rangle = 0 \text{ for } \omega_1 \neq \omega_2$$

   (a) For $v_a(t) = V_A \cos \omega t$ and $v_b(t) = V_B \cos(\omega t + \varphi)$, show that the mean-square sum is given by

$$\left\langle [v_a(t) + v_b(t)]^2 \right\rangle = \frac{1}{2}\left(V_A^2 + 2V_A V_B \cos\varphi + V_B^2\right)$$

   and the correlation coefficient is given by

$$\rho = \cos\varphi$$

   (b) What do the answers reduce to for $\varphi = 0$, $\varphi = \pm 180\,^\circ$, and $\varphi = \pm 90\,^\circ$? From the answers to this question, you should be able to conclude that two sinusoids of the same frequency are statistically independent if they differ in phase by $90\,^\circ$.

3. For $v_a(t) = V_A \cos \omega_1 t$ and $v_b(t) = V_B \cos(\omega_2 t + \varphi)$, where $\omega_1 \neq \omega_2$, show that

$$\left\langle [v_a(t) + v_b(t)]^2 \right\rangle = \frac{1}{2}\left(V_A^2 + V_B^2\right) \text{ and } \rho = 0$$

From this answer, you should be able to conclude that two sinusoids of differing frequencies are statistically independent regardless of the phase angle between the two.

4. For a complex impedance $Z = R + jX$, show that

$$|Z|^2 = Z \times Z^* = R^2 + X^2$$

$$\frac{1}{Z} = \frac{R - jX}{|Z|^2} \qquad \text{Re}\left(\frac{1}{Z}\right) = \frac{R}{|Z|^2} \qquad \text{Im}\left(\frac{1}{Z}\right) = -\frac{X}{|Z|^2}$$

5. An lossy inductor L has a series winding resistance R. The inductor can be modeled as an ideal inductor in series with a discrete resistor.

(a) Draw the circuit with a thermal phasor noise voltage V_t in series with the resistor. Calculate the short-circuit noise current $I_n = V_t/Z$, where Z is the complex impedance of the series circuit. Show that the mean-square short-circuit noise current is given by

$$i_n^2 = \overline{I_n I_n^*} = \frac{4kT\Delta f}{R} \frac{1}{1 + (2\pi f L/R)^2}$$

(b) Show that the answer above can be obtained from the expression

$$i_n^2 = 4kT \operatorname{Re}(Y) \Delta f$$

where Y is the complex admittance of the circuit. Note, this result can be thought of as the dual of the formula $v_n^2 = 4kT \operatorname{Re}(Z) \Delta f$ that was used in class to calculate the mean-square open-circuit voltage of a parallel RC network.

(c) Integrate the mean-square noise current to show that the total mean-square thermal noise current generated by the inductor, i.e. the noise in the band $0 \le f \le \infty$, is given by

$$i_{total}^2 = \frac{kT}{L}$$

Hint: Let $x = 2\pi f L/R$ and $df = (R/2\pi L)\, dx$. The integral which must be evaluated can be put into the form $\int_0^\infty dx/(1 + x^2) = [\tan^{-1} x]_0^\infty = \pi/2$.

6. A lossy inductor having an air core can be modeled as and ideal inductor L in series with a resistor R, where R is the winding resistance. Let a capacitor C be connected in parallel with the lossy inductor.

(a) With $s = j\omega$, where $\omega = 2\pi f$, solve for the complex impedance of the network. Use the general Nyquist equation $v_n^2 = 4kT \operatorname{Re}(Z) \Delta f$ to show that the mean-square thermal noise voltage across the circuit in the frequency band Δf is given by

$$v_n^2 = \frac{4kTR\Delta f}{(1 - \omega^2 LC)^2 + (\omega RC)^2}$$

(b) Replace the resistor with its Thévenin noise model. Use voltage division to show that the phasor noise voltage V_n across the circuit is given by

$$V_n = \frac{V_t}{1 - \omega^2 LC + j\omega RC}$$

where V_t is the thermal noise voltage generated by the resistor. Show that $v_n^2 = \overline{V_n V_n^*}$ gives the same answer as the one obtained above.

7. A resistor R and an ideal capacitor C are connected in parallel. The two are in thermal equilibrium. This means that the thermal noise power generated by the resistor that is absorbed by the capacitor must equal the thermal noise power generated by the capacitor that is absorbed by the resistor. Otherwise, one would be heating up while the other is cooling off and the two would not be in thermal equilibrium.

(a) Use the Thévenin noise model of the resistor and denote its thermal phasor noise voltage by V_t, where $v_t^2 = \overline{V_t V_t^*} = 4kTR\Delta f$ is the mean-square noise voltage in the band Δf at the frequency of analysis. Show that the phasor spot noise voltage V_n across the capacitor and phasor spot noise current I_n through the capacitor are given by

$$V_n = V_t \frac{1}{1 + j\omega RC} \qquad I_n = V_t \frac{j\omega C}{1 + j\omega RC}$$

(b) The power absorbed by the capacitor is given by $P_C = \operatorname{Re}\left(\overline{V_n I_n^*}\right)$. Show that this is zero. (Note, we are using the convention that the magnitude of a noise phasor is the rms value, not the peak value, so that there is no factor of 1/2 in the expression for P_C.)

(c) Because $P_C = 0$, it follows that the capacitor cannot absorb power from the resistor. How does this imply that the capacitor cannot generate noise power?

(d) Repeat the problem for an ideal inductor L in parallel with a resistor R. Show that V_n and I_n are given by

$$V_n = \frac{V_t}{R} \frac{j\omega L}{1 + j\omega L/R} \qquad I_n = \frac{V_t}{R} \frac{1}{1 + j\omega L/R}$$

8. A resistor R and a capacitor C are connected in parallel to form a two-terminal network. Use the Norton noise model of the resistor to show that the phasor short-circuit noise current $I_{n(sc)}$ and the phasor open-circuit noise voltage $V_{n(oc)}$ are given by

$$I_{n(sc)} = I_t \qquad V_{n(oc)} = \frac{I_t R}{1 + j\omega RC}$$

Show that the mean-square spot noise values are given by

$$i_{n(sc)}^2 = \frac{4kT}{R} \qquad v_{n(oc)}^2 = \frac{4kTR}{1 + (\omega RC)^2}$$

the complex correlation coefficient between $V_{n(oc)}$ and $I_{n(sc)}$ is given by

$$\gamma = \frac{\sqrt{1 + (\omega RC)^2}}{1 + j\omega RC}$$