Superposition Examples

The following examples illustrate the proper use of superposition of dependent sources. All superposition equations are written by inspection using voltage division, current division, series-parallel combinations, and Ohm’s law. In each case, it is simpler not to use superposition if the dependent sources remain active.

Example 1
The object is to solve for the current \( i \) in the circuit of Fig. 1. By superposition, one can write

\[
i = \frac{24}{3+2} - \frac{7}{3+2} \frac{2}{3+2} - \frac{3i}{3+2} = 2 - \frac{3}{5}i
\]

Solution for \( i \) yields

\[
i = \frac{2}{1+3/5} = \frac{5}{4} \text{ A}
\]

![Figure 1: Circuit for example 1.](image)

If superposition of the controlled source is not used, two solutions must be found. Let \( i = i_a + i_b \), where \( i_a \) is the current with the 7 A source zeroed and \( i_b \) is the current with the 24 V source zeroed. By superposition, we can write

\[
i_a = \frac{24}{3+2} - \frac{3i_a}{3+2} \quad i_b = -\frac{7}{3+2} \frac{2}{3+2} - \frac{3i_b}{3+2}
\]

Solution for \( i_a \) and \( i_b \) yields

\[
i_a = \frac{24}{3+2} \frac{1}{1 + \frac{3}{3+2}} = 3 \text{ A} \quad i_b = \frac{-7}{3+2} \frac{2}{1 + \frac{3}{3+2}} = -\frac{7}{4} \text{ A}
\]

The solution for \( i \) is thus

\[
i = i_a + i_b = \frac{5}{4} \text{ A}
\]

This is the same answer obtained by using superposition of the controlled source.

Example 2
The object is to solve for the voltages \( v_1 \) and \( v_2 \) across the current sources in Fig. 2, where the datum node is the lower branch. By superposition, the current \( i \) is given by

\[
i = 2 \frac{7}{7+15+5} + \frac{3}{7+15+5} + 4i \frac{7+15}{7+15+5} = \frac{17}{27} + \frac{88i}{27}
\]

Solution for \( i \) yields

\[
i = \frac{17/27}{1 - 88/27} = -\frac{17}{61} \text{ A}
\]

Although superposition can be used to solve for \( v_1 \) and \( v_2 \), it is simpler to write

\[
v_2 = 5i = -1.393 \text{ V} \quad v_1 = v_2 - (4i - i) 15 = 11.148 \text{ V}
\]
Example 3

The object is to solve for the current $i_1$ in the circuit of Fig. 3. By superposition, one can write

$$i_1 = \frac{30}{6+4+2} + \frac{4}{6+4+2} - \frac{8i_1}{6+4+2} - \frac{6}{6+4+2} = \frac{42}{12} - 4i_1$$

Solution for $i_1$ yields

$$i_1 = \frac{42/12}{1+4} = 0.7 \text{ A}$$

Example 4

The object is to solve for the Thévenin equivalent circuit seen looking into the terminals $A - A'$ in the circuit of Fig. 4. By superposition, the voltage $v_x$ is given by

$$v_x = (3 - i_o) (2||40) + 5v_x \frac{2}{40+2} = \frac{80}{42} (3 - i_o) + \frac{10}{42} v_x$$

where $i_o$ is the current drawn by any external load and the symbol "||" denotes a parallel combination. Solution for $v_x$ yields

$$v_x = \frac{80/42}{1-10/42} (3 - i_o) = 2.5 (3 - i_o)$$

Although superposition can be used to solve for $v_o$, it is simpler to write

$$v_o = v_x - 5v_x = -30 + 10i_o$$

It follows that the Thévenin equivalent circuit consists of a $-30 \text{ V}$ source in series with a $-10 \Omega$ resistor. The circuit is shown in Fig. 5.
Example 5

The object is to solve for the voltage $v_o$ in the circuit of Fig. 6. By superposition, the current $i_b$ is given by

$$i_b = \frac{70}{4\parallel20+2\parallel10} + \frac{20}{4+20} + \frac{50}{10+4\parallel20\parallel2} - \frac{20\parallel2}{4+20\parallel2} + \frac{2i_b}{20\parallel2+4\parallel10} = \frac{35}{3} + \frac{25}{18} - \frac{11}{36}i_b$$

Solution for $i_b$ yields

$$i_b = \frac{35/3 + 25/18}{1 + 11/36} = 10\,\text{A}$$

Although superposition can be used to solve for $v_o$, it is simpler to write

$$v_o = 70 - 4i_b = 30\,\text{V}$$
Example 6
The object is to solve for the voltage $v_o$ in the circuit of Fig. 7. By superposition, the voltage $v_\Delta$ is given by

$$v_\Delta = -0.4v_\Delta \times 10 + 5 \times 10$$

This can be solved for $v_\Delta$ to obtain

$$v_\Delta = \frac{5 \times 10}{1 + 0.4 \times 10} = 10 \text{ V}$$

By superposition, $i_\Delta$ is given by

$$i_\Delta = \frac{10}{5 + 20} - 0.4v_\Delta \frac{20}{20 + 5} = \frac{10}{25} - 0.4v_\Delta \frac{20}{25} = -\frac{70}{25} \text{ A}$$

Thus $v_o$ is given by

$$v_o = 10 - 5i_\Delta = 24 \text{ V}$$

![Figure 7: Circuit for example 6.](image)

Example 7
The object is to solve for the voltage $v$ as a function of $v_s$ and $i_s$ in the circuit in Fig. 8. By superposition, the current $i$ is given by

$$i = \frac{v_s}{5} - \frac{2}{5}i_s - \frac{3}{5} \times 3i$$

This can be solved for $i$ to obtain

$$i = \frac{v_s}{14} - i_s$$

By superposition, the voltage $v$ is given by

$$v = \frac{v_s}{5} - \frac{2}{5}i_s + \frac{2}{5} \times 3i$$

$$= \frac{v_s}{5} - \frac{2}{5}i_s + \frac{2}{5} \times 3 \left( \frac{v_s}{14} - \frac{i_s}{7} \right)$$

$$= \frac{2}{7}v_s - \frac{4}{7}i_s$$

Example 8
This example illustrates the use of superposition in solving for the dc bias currents in a BJT. The object is to solve for the collector current $I_C$ in the circuit of Fig. 9. Although no explicit dependent sources are shown, the three BJT currents are related by $I_C = \beta I_B = aI_E$, where $\beta$ is the current gain and $a = \beta/(1 + \beta)$. If any one of the currents is zero, the other two must also be zero. However, the currents can be treated as independent variables in using superposition.
Figure 8: Circuit for Example 7.

Figure 9: Circuit for example 8.
By superposition of $V^+$, $I_B = I_C/\beta$, and $I_C$, the voltage $V_B$ is given by

$$V_B = V^+ \frac{R_2}{R_C + R_1 + R_2} - \frac{I_C}{\beta} \left[ \frac{1}{(R_C + R_1) || R_2} \right]$$

$$\quad - \frac{I_C}{R_C + R_1 + R_2}$$

A node-voltage solution for $V_B$ requires the solution of two simultaneous equations to obtain the same answer which superposition yields by inspection. This equation and the equation

$$V_B = V_{BE} + \frac{I_C}{\alpha} R_E$$

can be solved for $I_C$ to obtain

$$I_C = \frac{V^+ \frac{R_2}{(R_C + R_1) || R_2} - V_{BE}}{\frac{1}{\beta} \left[ \frac{1}{(R_C + R_1) || R_2} \right] + \frac{R_C R_2}{R_C + R_1 + R_2} + \frac{R_E}{\alpha}}$$

In most contemporary electronics texts, the value $V_{BE} = 0.7 \text{V}$ is assumed in BJT bias calculations.

**Example 9**

This example illustrates the use of superposition to solve for the small-signal base input resistance of a BJT. Fig. 10 shows the small-signal BJT hybrid-pi model with a resistor $R_E$ from emitter to ground and a resistor $R_C$ from collector to ground. In the model, $r_\pi = V_T/I_B$ and $r_0 = (V_A + V_{CE})/I_C$, where $V_T$ is the thermal voltage, $I_B$ is the dc base current, $V_A$ is the Early voltage, $V_{CE}$ is the dc collector-emitter voltage, and $I_C$ is the dc collector current.

![Figure 10: Circuit for example 9.](image)

By superposition of $i_b$ and $\beta i_b$, the base voltage $v_b$ is given by

$$v_b = i_b \left[ r_\pi + \frac{R_E}{(r_0 + R_C)} \right] + \beta i_b \frac{r_0}{R_E + r_0 + R_C} R_E$$

This can be solved for the base input resistance $r_{ib} = v_b/i_b$ to obtain

$$r_{ib} = r_\pi + \frac{\beta r_0 R_E}{R_E + r_0 + R_C}$$

which simplifies to

$$r_{ib} = r_\pi + \frac{R_E}{R_E + r_0 + R_C} \left( 1 + \beta \right) r_0 + R_C$$

A node-voltage solution for $r_{ib}$ requires the solution of three simultaneous equations to obtain the same answer which follows almost trivially by superposition.
Example 10

This example illustrates the use of superposition with an op-amp circuit. The circuit is shown in Fig. 11. The object is to solve for $v_O$. With $v_2 = 0$, it follows that $v_A = v_1$, $v_B = 0$, and $v_C = [1 + R_4/(R_3||R_5)] v_1$. By superposition of $v_A$ and $v_C$, $v_O$ can be written

$$v_O = -\frac{R_2}{R_5} v_A - \frac{R_2}{R_1} v_C = - \left[ \frac{R_2}{R_5} + \frac{R_2}{R_1} \left( 1 + \frac{R_4}{R_3||R_5} \right) \right] v_1$$

With $v_1 = 0$, it follows that $v_A = 0$, $v_B = v_2$, and $v_C = -(R_4/R_5) v_2$. By superposition of $v_2$ and $v_C$, $v_O$ can be written

$$v_O = \left( 1 + \frac{R_2}{R_1||R_5} \right) v_2 - \frac{R_2}{R_1} v_C$$

$$= \left( 1 + \frac{R_2}{R_1||R_5} + \frac{R_2 R_4}{R_1 R_5} \right) v_2$$

Thus the total expression for $v_O$ is

$$v_O = - \left[ \frac{R_2}{R_5} + \frac{R_2}{R_1} \left( 1 + \frac{R_4}{R_3||R_5} \right) \right] v_1$$

$$+ \left( 1 + \frac{R_2}{R_1||R_5} + \frac{R_2 R_4}{R_1 R_5} \right) v_2$$

![Figure 11: Circuit for Example 10.](image)

Example 11

Figure 12 shows a circuit that might be encountered in the noise analysis of amplifiers. The amplifier is modeled by a $z$-parameter model. The square sources represent noise sources. $V_{ts}$ and $I_{tA}$, respectively, model the thermal noise generated by $Z_x$ and $Z_A$. $V_n$ and $I_n$ model the noise generated by the amplifier. The amplifier load is an open circuit so that $I_2 = 0$. The open-circuit output voltage is given by

$$V_{o(oc)} = z_{12} I_1 + I_A Z_A$$

By superposition, the currents $I_1$ and $I_A$ are given by

$$I_1 = \frac{V_s + V_{ts} + V_n}{Z_s + Z_A + z_{11}} + \frac{I_n}{Z_s + Z_A + z_{11}} \frac{Z_A}{Z_s + Z_A + z_{11}}$$

$$- I_{tA} \frac{Z_A}{Z_s + Z_A + z_{11}}$$

$$I_A = \frac{V_s + V_{ts} + V_n}{Z_s + Z_A + z_{11}} - \frac{I_n}{Z_s + Z_A + z_{11}} \frac{z_{11}}{Z_s + Z_A + z_{11}}$$

$$+ I_{tA} \frac{Z_s + z_{11}}{Z_s + Z_A + z_{11}}$$

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Note that when $I_n = 0$, the sources $V_s$, $V_{ts}$, and $V_n$ are in series and can be considered to be one source equal to the sum of the three. When these are substituted into the equation for $V_{o(oc)}$ and the equation is simplified, we obtain

$$V_{o(oc)} = \frac{z_{21} + Z_A}{Z_S + Z_A + z_{11}} \left[ V_s + V_{ts} + V_n ight. \\
+ I_n (Z_S + Z_A) z_{21} - Z_A z_{11} \\
\left. + I_{tA} \frac{Z_A z_{21} - (Z_S + z_{11}) Z_A}{z_{21} + Z_A} \right]$$

Figure 12: Circuit for Example 11.

**Example 12**

It is commonly believed that superposition can only be used with circuits that have more than one source. This example illustrates how it can be used with a circuit having one. Consider the first-order all-pass filter shown in Fig. 13(a). An equivalent circuit is shown in Fig. 13(b) in which superposition can be used to write by inspection

$$V_o = \left(1 + \frac{R_1}{R_I}\right) \frac{RC_S}{1 + RC_S} V_i - \frac{R_1}{R_I} V_i = \frac{RC_S - 1}{RC_S + 1} V_i$$
Figure 13: Circuit for Example 12.