Transfer Functions and Bode Plots

Transfer Functions

For sinusoidal time variations, the input voltage to a filter can be written

$$v_I(t) = \mathrm{Re}\left[V_i e^{j\omega t}\right]$$

where V_i is the phasor input voltage, i.e. it has an amplitude and a phase, and $e^{j\omega t} = \cos\omega t + j\sin\omega t$. A sinusoidal signal is the only signal in nature that is preserved by a linear system. Therefore, if the filter is linear, its output voltage can be written

$$v_O(t) = \mathrm{Re}\left[V_o e^{j\omega t}\right]$$

where V_o is the phasor output voltage. The ratio of V_o to V_i is called the voltage-gain transfer function. It is a function of frequency. Let us denote

$$T(j\omega) = \frac{V_o}{V_i}$$

We can write $T(j\omega)$ as follows:

$$T(j\omega) = A(\omega)\, e^{j\varphi(\omega)}$$

where $A(\omega)$ and $\varphi(\omega)$ are real functions of ω. $A(\omega)$ is called the gain function and $\varphi(\omega)$ is called the phase function.

As an example, consider the filter input voltage

$$v_I(t) = V_1\cos(\omega t + \theta) = \mathrm{Re}\left[V_1 e^{j\theta} e^{j\omega t}\right]$$

The corresponding phasor input and output voltages are

$$V_i = V_1 e^{j\theta}$$

$$V_o = V_1 e^{j\theta} A(\omega)\, e^{j\varphi(\omega)}$$

It follows that the time domain output voltage is

$$v_O(t) = \mathrm{Re}\left[V_1 e^{j\theta} A(\omega)\, e^{j\varphi(\omega)} e^{j\omega t}\right] = A(\omega)\, V_1 \cos\left[\omega t + \theta + \varphi(\omega)\right]$$

This equation illustrates why $A(\omega)$ is called the gain function and $\varphi(\omega)$ is called the phase function.

The complex frequency s is usually used in place of $j\omega$ in writing transfer functions. In general, most transfer functions can be written in the form

$$T(s) = K\frac{N(s)}{D(s)}$$

where K is a gain constant and $N(s)$ and $D(s)$ are polynomials in s containing no reciprocal powers of s. The roots of $D(s)$ are called the poles of the transfer function. The roots of $N(s)$ are called the zeros.

As an example, consider the function

$$T(s) = 4\frac{s/4+1}{s^2/6+5s/6+1} = 4\frac{s/4+1}{(s/2+1)(s/3+1)}$$

The function has a zero at $s = -4$ and poles at $s = -2$ and $s = -3$. Note that $T(\infty) = 0$. Because of this, some texts would say that $T(s)$ has a zero at $s = \infty$. However, this is not correct because $N(\infty) \neq 0$.

Note that the constant terms in the numerator and denominator of $T(s)$ are both unity. This is one of two standard ways for writing transfer functions. Another way is to make the coefficient of the highest powers of s unity. In this case, the above transfer function would be written

$$T(s) = 6\frac{s+4}{s^2+5s+6} = 6\frac{s+4}{(s+2)(s+3)}$$

Because it is usually easier to construct Bode plots with the first form, that form is used here.

Because the complex frequency s is the operator which represents d/dt in the differential equation for a system, the transfer function contains the differential equation. Let the transfer function above represent the voltage gain of a circuit, i.e. $T(s) = V_o/V_i$, where V_o and V_i, respectively, are the phasor output and input voltages. It follows that

$$\left(\frac{s^2}{6} + \frac{5s}{6} + 1\right) V_o = 4\left(\frac{s}{4} + 1\right) V_i$$

When the operator s is replaced with d/dt, the following differential equation is obtained:

$$\frac{1}{6}\frac{d^2 v_O}{dt^2} + \frac{5}{6}\frac{dv_O}{dt} + v_O = \frac{dv_I}{dt} + 4v_I$$

where v_O and v_I, respectively, are the time domain output and input voltages. Note that the poles are related to the derivatives of the output and the zeros are related to the derivatives of the input.

How to Construct Bode Plots

A Bode plot is a plot of either the magnitude or the phase of a transfer function $T(j\omega)$ as a function of ω. The magnitude plot is the more common plot because it represents the gain of the system. Therefore, the term "Bode plot" usually refers to the magnitude plot. The rules for making Bode plots can be derived from the following transfer function:

$$T(s) = K\left(\frac{s}{\omega_0}\right)^{\pm n}$$

where n is a positive integer. For $+n$ as the exponent, the function has n zeros at $s = 0$. For -n, it has n poles at $s = 0$. With $s = j\omega$, it follows that $T(j\omega) = Kj^{\pm n}(\omega/\omega_0)^{\pm n}$, $|T(j\omega)| = K(\omega/\omega_0)^{\pm n}$ and $\angle T(j\omega) = \pm n \times 90\,^\circ$. If ω is increased by a factor of 10, $|T(j\omega)|$ changes by a factor of $10^{\pm n}$. Thus a plot of $|T(j\omega)|$ versus ω on log–log scales has a slope of $\log(10^{\pm n}) = \pm n$ decades/decade. There are 20 dBs in a decade, so the slope can also be expressed as $\pm 20n$ dB/decade.

As a first example, consider the low-pass transfer function

$$T(s) = \frac{K}{1 + s/\omega_1}$$

This function has a pole at $s = -\omega_1$ and no zeros. For $s = j\omega$ and $\omega/\omega_1 \ll 1$,we have $T(j\omega) \simeq K$, $|T(j\omega)| \simeq K$, and $\angle T(j\omega) \simeq 0 \times 90\,^\circ = 0\,^\circ$. For $\omega/\omega_1 \gg 1$, $T(j\omega) \simeq K(j\omega/\omega_1)^{-1}$, $|T(j\omega)| \simeq K(\omega/\omega_1)^{-1}$, and $\angle T(j\omega) \simeq -1{\times}90\,^\circ = -90\,^\circ$. On $\log - \log$ scales, the magnitude plot for the low-frequency approximation has a slope of 0 while that for the high-frequency approximation has a slope of -1. The low and high-frequency approximations intersect when $K = K(\omega_1/\omega)$, or when $\omega = \omega_1$. For $\omega = \omega_1$, $|T(j\omega)| = K/|1+j| = K/\sqrt{2}$ and $\angle T(j\omega) = -\arctan(1) = -45\,^\circ$. Note that this is the average value of the phase on the two adjoining asymptotes. The Bode magnitude and phase plots are shown in Fig. 1. Note that the slope of the asymptotic magnitude plot rotates by -1 at $\omega = \omega_1$. Because ω_1 is the magnitude of the pole frequency, we say that the slope rotates by -1 at a pole. A straight line segment that is tangent to the phase plot at $\omega = \omega_1$ would intersect the $0\,^\circ$ level at $\omega_1/4.81$ and the $-90\,^\circ$ level at $4.81\omega_1$.

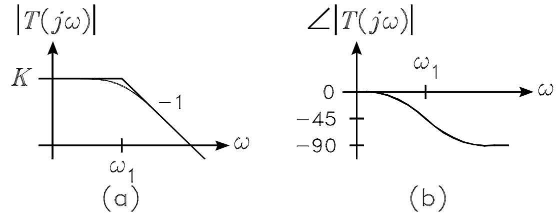

Figure 1: Bode plots. (a) Magnitude. (b) Phase.

As a second example, consider the transfer function

$$T(s) = K\left(1 + \frac{s}{\omega_1}\right)$$

This function has a zero at $s = -\omega_1$. For $s = j\omega$ and $\omega/\omega_1 \ll 1$,we have $T(j\omega) \simeq K$, $|T(j\omega)| \simeq K$, and $\angle T(j\omega) \simeq 0 \times 90\,^\circ = 0\,^\circ$. For $\omega/\omega_1 \gg 1$, $T(j\omega) \simeq K(j\omega/\omega_1)^1$ $|T(j\omega)| \simeq K(\omega/\omega_1)$ and $\angle T(j\omega) \simeq +1 \times 90\,^\circ = 90\,^\circ$. On $\log - \log$ scales, the magnitude plot for the low-frequency approximation has a slope of 0 while that for the high-frequency approximation has a slope of $+1$. The low and high-frequency approximations intersect when $K = K(\omega/\omega_1)$, or when $\omega = \omega_1$. For $\omega = \omega_1$, $|T(j\omega)| = K\sqrt{2}$ and $\angle T(j\omega) = \arctan(1) = 45\,^\circ$. Note that this is the average of the phase on the two adjoining asymptotes. The Bode magnitude and phase plots are shown in Fig. 2. Note that the slope of the asymptotic magnitude plot rotates by $+1$ at $\omega = \omega_1$. Because ω_1 is the magnitude of the zero frequency, we say that the slope rotates by $+1$ at a zero. A straight line segment that is tangent to the phase plot at $\omega = \omega_1$ would intersect the $0\,^\circ$ level at $\omega_1/4.81$ and the $90\,^\circ$ level at $4.81\omega_1$.

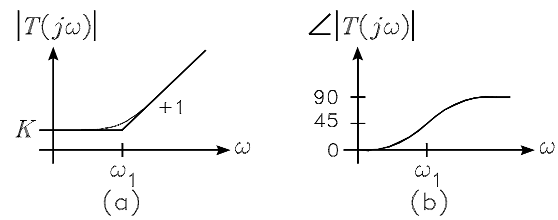

Figure 2: Bode plots. (a) Magnitude. (b) Phase.

From the above examples, we can summarize the basic rules for making Bode plots as follows:

1. In any frequency band where a transfer function can be approximated by $K\left(j\omega/\omega_0\right)^{\pm n}$, the slope of the Bode magnitude plot is $\pm n$ dec/dec. The phase is $\pm n \times 90\,^\circ$.
2. Poles cause the asymptotic slope of the magnitude plot to rotate clockwise by one unit at the pole frequency.
3. Zeros cause the asymptotic slope of the magnitude plot to rotate counter-clockwise by one unit at the zero frequency.

As a third example, consider the transfer function

$$T\left(s\right) = K\frac{s/\omega_1}{s/\omega_1 + 1}$$

This function has a pole at $s = -\omega_1$ and a zero at $s = 0$. For $s = j\omega$ and $\omega/\omega_1 \ll 1$,we have $\left|T\left(j\omega\right)\right| \simeq K\left(\omega/\omega_1\right)$ and $\angle T\left(j\omega\right) \simeq 90\,^\circ$. For $\omega/\omega_1 \gg 1$, $\left|T\left(j\omega\right)\right| \simeq K$ and $\angle T\left(j\omega\right) \simeq 0\,^\circ$. On $\log - \log$ scales, the magnitude plot for the low-frequency approximation has a slope of $+1$ while that for the high-frequency approximation has a slope of 0. The low and high-frequency approximations intersect when $K\left(\omega/\omega_1\right) = K$, or when $\omega = \omega_1$. For $\omega = \omega_1$, $\left|T\left(j\omega\right)\right| = K/\sqrt{2}$ and $\angle T\left(j\omega\right) = 90^{\text{o}} - \arctan\left(1\right) = 45\,^\circ$. The Bode magnitude and phase plots are shown in Fig. 3. Note that the slope of the asymptotic magnitude plot rotates by -1 at the pole. The transfer function is called a high-pass function because its gain approaches zero at low frequencies.

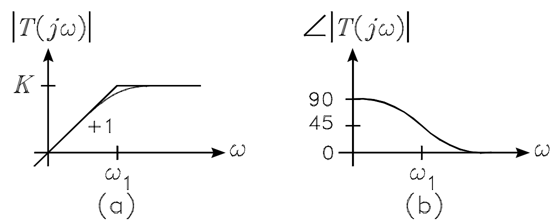

Figure 3: Bode plots. (a) Magnitude. (b) Phase.

A shelving transfer function has the form

$$T\left(s\right) = K\frac{1 + s/\omega_2}{1 + s/\omega_1}$$

The function has a pole at $s = -\omega_1$ and a zero at $s = -\omega_2$. We will consider the low-pass shelving function for which $\omega_1 < \omega_2$. For $s = j\omega$ and $\omega/\omega_1 \ll 1$, we have $\left|T\left(j\omega\right)\right| \simeq K$ and $\angle T\left(j\omega\right) \simeq 0\,^\circ$. As ω is increased, the pole causes the asymptotic slope to rotate from 0 to -1 at ω_1. The zero causes the asymptotic slope to rotate from -1 back to 0 at ω_2. For $\omega/\omega_2 \gg 1$, $\left|T\left(j\omega\right)\right| \simeq K\left(\omega_1/\omega_2\right)$. The Bode magnitude plot is shown in Fig. 4(a). If the transfer function did not have the zero, the actual gain at ω_1 would be $K/\sqrt{2}$. The zero causes the gain to be between $K/\sqrt{2}$ and K. Similarly, the pole causes the actual gain at ω_2 to be between $K\left(\omega_1/\omega_2\right)$ and $\sqrt{2}K\left(\omega_1/\omega_2\right)$. The actual plot intersects the asymptotic plot at the geometric mean frequency $\sqrt{\omega_1\omega_2}$.

The phase plot has a slope that approaches $0°$ at very low frequencies and at very high frequencies. At the geometric mean frequency $\sqrt{\omega_1\omega_2}$, the phase is approaching $-90°$. If the function only had a pole, the phase at ω_1 would be $-45°$, approaching $-90°$ at higher frequencies. However, the zero causes the high-frequency phase to approach $0°$. Thus the phase at ω_1 is more positive than $-45°$. At the geometric mean frequency $\sqrt{\omega_1\omega_2}$, the slope of the phase function is zero. The Bode phase plot is shown in Fig. 4(b).

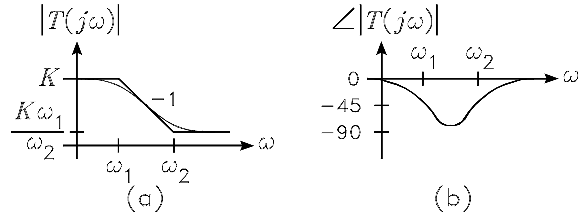

Figure 4: Bode plots. (a) Magnitude. (b) Phase.

Impedance Transfer Functions

RC Network

The impedance transfer function for a two-terminal *RC* network which contains only one capacitor and is not an open circuit at dc can be written

$$Z = R_{dc}\frac{1+\tau_z s}{1+\tau_p s}$$

where R_{dc} is the dc resistance of the network, τ_p is the pole time constant, and τ_z is the zero time constant. The pole time constant is the time constant of the network with the terminals open circuited. The zero time constant is the time constant of the network with the terminals short circuited. Figure 5(a) shows the circuit diagram of an example two-terminal *RC* network. The impedance transfer function can be written by inspection to obtain

$$Z = R_1\frac{1+R_2Cs}{1+(R_1+R_2)\,Cs}$$

Figure 5: Example *RC* and *RL* impedance networks.

RL Network

The impedance transfer function for a two-terminal RL network which contains only one inductor and is not a short circuit at dc can be written

$$Z = R_{dc}\frac{1+\tau_z s}{1+\tau_p s}$$

where R_{dc} is the dc resistance of the network, τ_p is the pole time constant, and τ_z is the zero time constant. The pole time constant is the time constant of the network with the terminals open circuited. The zero time constant is the time constant of the network with the terminals short circuited. Figure 5(b) shows the circuit diagram of an example two-terminal RL network. The impedance transfer function can be written by inspection to obtain

$$Z = R_1 \| R_2 \frac{1+(L/R_2)\,s}{1+[L/(R_1+R_2)]\,s}$$

Voltage Divider Transfer Functions

RC Network

The voltage-gain transfer function of a RC voltage-divider network containing only one capacitor and having a non-zero gain at dc can be written

$$\frac{V_o}{V_i} = K_{dc}\frac{1+\tau_z s}{1+\tau_p s}$$

where K_{dc} is the dc gain (C an open circuit), τ_p is the pole time constant, and τ_z is the zero time constant. The pole time constant is the time constant of the network with $V_i = 0$ and V_o open circuited. The zero time constant is the time constant of the network with $V_o = 0$ and V_i open circuited. Figure 6(a) shows the circuit diagram of an example RC network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_2+R_3}{R_1+R_2+R_3} \times \frac{1+(R_2\|R_3)\,Cs}{1+[(R_1+R_2)\,\|R_3]\,Cs}$$

Figure 6(b) shows the circuit diagram of a second example RC network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_3}{R_1+R_3} \times \frac{1+(R_1+R_2)\,Cs}{1+[(R_1\|R_3)+R_2]\,Cs}$$

High-Pass RC Network

The voltage-gain transfer function of a high-pass RC voltage-divider network containing only one capacitor can be written

$$\frac{V_o}{V_i} = K_\infty\frac{\tau_p s}{1+\tau_p s}$$

where K_∞ is the infinite frequency gain (C a short circuit) and τ_p is the pole time constant. The pole time constant is calculated with $V_i = 0$ and V_o open circuited. Figure 6(c) shows the circuit diagram of a third example RC network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_2}{R_1+R_2} \times \frac{(R_1+R_2)\,Cs}{1+(R_1+R_2)\,Cs}$$

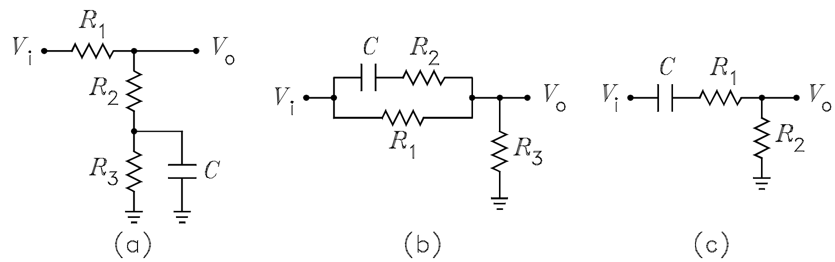

Figure 6: Example *RC* voltage divider networks.

RL Network

The voltage-gain transfer function of a *RL* voltage-divider network containing only one inductor and having a non-zero gain at dc can be written

$$\frac{V_o}{V_i} = K_{dc}\frac{1+\tau_z s}{1+\tau_p s}$$

where K_∞ is the zero frequency gain (L a short circuit), τ_p is the pole time constant, and τ_z is the zero time constant. The pole time constant is the time constant of the network with $V_i = 0$ and V_o open circuited. The zero time constant is the time constant of the network with $V_o = 0$ and V_i open circuited. Figure 7(a) shows the circuit diagram of an example *RL* network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_2}{R_1+R_2} \times \frac{1+[L/(R_2\|R_3)]\,s}{1+(L/[(R_1+R_2)\|R_3])\,s}$$

Figure 7(b) shows the circuit diagram of a second example *RL* network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_3}{R_1+R_3} \times \frac{1+(L/R_1)\,s}{1+(L/[R_1\|(R_2+R_3)])\,s}$$

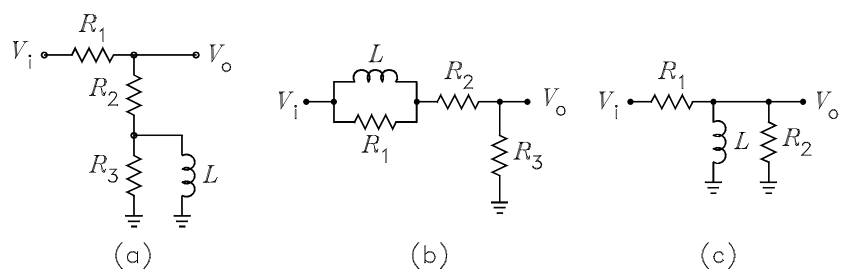

Figure 7: Example *RL* voltage divider circuits.

High-Pass RL Network

The voltage-gain transfer function of a high-pass RL voltage-divider network containing only one inductor can be written

$$\frac{V_o}{V_i} = K_\infty \frac{\tau_p s}{1 + \tau_p s}$$

where K_∞ is the infinite frequency gain (L an open circuit) and τ_p is the pole time constant. The pole time constant is calculated with $V_i = 0$ and V_o open circuited. Figure 7(c) shows the circuit diagram of a third example RL network. The voltage-gain transfer function can be written by inspection to obtain

$$\frac{V_o}{V_i} = \frac{R_2}{R_1 + R_2} \times \frac{\left[L/\left(R_1 \| R_2\right)\right] s}{1 + \left[L/\left(R_1 \| R_2\right)\right] s}$$